THE NINTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Ninth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Monday and Tuesday, December 29–30, 1902. The following sixty-two members were in attendance at the four sessions:

Dr. Grace Andrews, Mr. C. H. Ashton, Professor Joseph Bowden, Professor E. W. Brown, Dr. A. B. Chace, Professor A. S. Chessin, Dr. J. E. Clarke, Dr. A. B. Coble, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Mr. D. R. Curtiss, Dr. W. S. Dennett, Dr. Otto Dunkel, Professor T. C. Esty, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Dr. G. B. Germann, Dr. G. H. Hallett, Miss Carrie Hammerslough, Professor James Harkness, Dr. H. E. Hawkes, Dr. E. R. Hedrick, Dr. A. A. Himowich, Dr. E. V. Huntington, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Professor Pomeroy Ladue, Professor P. A. Lambert, Professor J. L. Love, Dr. Emory McClintock, Professor H. P. Manning, Professor W. H. Metzler, Mr. H. B. Mitchell, Professor E. H. Moore, Professor F. Morley, Professor G. D. Olds, Professor W. F. Osgood, Professor Anna H. Palmié, Professor James Pierpont, Professor R. W. Prentiss, Professor M. I. Pupin, Dr. I. E. Rabinovitch, Professor E. D. Roe, Jr., Professor Charlotte A. Scott, Mr. Burke Smith, Professor D. E. Smith, Professor P. F. Smith, Dr. H. F. Stecker, Professor Irving Stringham, Dr. W. M. Strong, Professor H. W. Tyler, Professor E. B. Van Vleck, Dr. Roxana H. Vivian, Professor L. A. Wait, Mr. A. C. Washburne, Professor F. S. Woods, Professor R. S. Woodward, Professor J. W. A. Young, Mr. J. W. Young.

The retiring President of the Society, Professor Eliakim Hastings Moore, presided at the earlier sessions, being relieved by Vice-Presidents Professor F. Morley and W. F. Osgood, and giving way on Tuesday to the President-Elect, Professor Thomas Scott Fiske. The Council announced the election of the following persons to membership in the Society: Dr. A. B. Coble, University of Missouri; Mr. W. A. Cornish, State Normal School, Cortland, N. Y.; Dr. A. G. Hall, University

of Michigan; Mr. E. L. Hancock, Purdue University; Professor L. M. Hoskins, Stanford University; Mr. W. D. A. Westfall, Yale University; Mr. W. F. White, State Normal School, New Paltz, N. Y. Sixteen applications for admission to the Society were received.

Reports were presented by the Librarian, the Treasurer, and the Auditing Committee. These reports will appear in the Annual Register, now in press. During the past year the membership of the Society has increased from 375 to 399; the number of papers presented at the meetings from 133 to 156. The total attendance at all meetings was 266; 144 members attended at least one meeting. The library now contains about 1000 bound volumes, and nearly 120 mathematical journals are regularly received in exchange for the *Bulletin* and the *Transactions*. A catalogue of the library will be published in the Annual Register.

Coming at the expiration of the presidential term of office, the meeting was especially marked as the occasion of a presidential address. Under the title: "On the foundations of mathematics," the retiring President, Professor E. H. Moore, urged the desirability of the Society undertaking to exert an effective influence on the teaching of elementary mathematics. A committee of the Council was appointed to consider this question.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen:

President,	Professor THOMAS S. FISKE.
Vice-Presidents,	Professor W. F. OSGOOD.
	Professor ALEXANDER ZIWET.
Secretary,	Professor F. N. COLE.
Treasurer,	DR. W. S. DENNETT.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1905,

Professor JAMES HARKNESS, Professor IRVING STRINGHAM,
Professor HEINRICH MASCHKE, Professor H. W. TYLER.

The following papers were read at the annual meeting:

(1) Dr. E. V. HUNTINGTON: "A complete set of postulates for the theory of real numbers (second paper)."

(2) Dr. E. V. HUNTINGTON: "On the definitions of the elementary functions by means of definite integrals."

(3) Dr. C. J. KEYSER: "On the axiom of infinity."

(4) Professor G. H. DARWIN: "The approximate determination of the form of Maclaurin's spheroid."

(5) Professor HARRIS HANCOCK: "Remarks on the sufficient conditions in the calculus of variations."

(6) Professor L. E. DICKSON: "The abstract group G simply isomorphic with the alternating group on six letters."

(7) Professor E. H. MOORE: Presidential address: "On the foundations of mathematics."

(8) Professor W. E. TAYLOR: "On the product of an alternant and a symmetric function."

(9) Professor E. D. ROE, JR.: "On the coefficients in the product of an alternant and a symmetric function."

(10) Professor E. D. ROE, JR.: "On the coefficients in the quotient of two alternants" (preliminary communication).

(11) Professor E. O. LOVETT: "A transformation group of $(2n-1)(n-1)$ parameters, and its rôle in the problem of n bodies."

(12) Dr. I. E. RABINOVITCH: "On the solid lunes of conicoids, analogous to the circular lunes of Hippocrates of Chios."

(13) Dr. E. B. WILSON: "The synthetic treatment of conics at the present time."

(14) Dr. A. B. COBLE: "On the invariant theory of the connex (2, 2) of the ternary domain viewed as a connex (1, 1) in a five dimensional space."

(15) Dr. EDWARD KASNER: "The general quadratic systems of conics and quadrics."

(16) Professor W. F. OSGOOD: On the transformation of the boundary in the case of conformal mapping."

(17) Professor W. F. OSGOOD: "A Jordan curve of positive area."

(18) Mr. J. W. YOUNG: "On the automorphic functions associated with the group of character $[0, 3; 2, 4, \infty]$" (preliminary report).

(19) Mr. R. W. H. T. HUDSON: "The analytic theory of displacements."

(20) Dr. H. E. HAWKES: "Enumeration of the non-quaternion number systems."

(21) Dr. H. F. STECKER: "On the parameters in certain systems of geodesic lines."

(22) Dr. EDWARD KASNER: "The generalized Beltrami problem concerning geodesic representation."

(23) Professor MAXIME BÔCHER: "Singular points of functions which satisfy partial differential equations of the elliptic type."

(24) Mr. G. D. BIRKHOFF and Mr. H. S. VANDIVER: "General theory of the integral divisors of $a^n - b^n$, and allied cyclotomic forms."

(25) Professor G. A. MILLER: "A new proof of the generalized Wilson theorem."

(26) Professor F. MORLEY: "On the determinant $|(x_i - a_j)^{-2}|$."

Professor Darwin's paper was communicated to the Society through Professor E. W. Brown, Professor Taylor's through Professor Metzler, and Mr. Hudson's through Professor Morley. Mr. Vandiver was introduced by Professor Crawley. In the absence of the authors, Professor Darwin's paper was read by Professor Brown, Professor Taylor's by Professor Roe, Professor Bôcher's by Professor Osgood, and the papers of Professor Hancock, Professor Dickson, Professor Lovett, Dr. Wilson, Mr. Hudson and Professor Miller were read by title.

Dr. Wilson's paper and Professor Osgood's first paper appeared in the February BULLETIN. The papers of Professor Dickson and Mr. Hudson are contained in the present number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. The two sets of postulates for real number * contained in Dr. Huntington's present paper are more convenient and natural than the set proposed by him at the April meeting (see abstract in BULLETIN, volume 8, page 371). The fundamental concepts involved in the present paper are 1° a *relation*, $\odot$, in which one element may stand to another — $a \odot b$ being read: a "less than" b; 2° a *rule of combination*, $\oplus$, according to which every two elements a and b of a given assemblage de-

*Cf. D. Hilbert, *Jahresber. d. Deutschen Math.-Vereinigung*, volume 8 (1900), page 180.

termine uniquely an object $a \oplus b$ called their "sum"; and 3° another *rule of combination*, $\odot$, according to which a and b determine uniquely an object $a \odot b$ called their "product." Any assemblage in which $\otimes$ and $\oplus$ are so defined as to satisfy the nine postulates of the first set below, or any assemblage in which $\otimes$, $\oplus$ and $\odot$ are so defined as to satisfy the thirteen postulates of the second set, will be equivalent to the system of all real numbers (positive, negative and zero). Each set is "complete"; that is, the postulates of each set are (*a*) consistent, (*b*) independent, and (*c*) sufficient to define essentially a single assemblage.

The first set contains the following postulates:

1. If $a \neq b$ then either $a \otimes b$ or $b \otimes a$.
2. If $a \otimes b$ and $b \otimes c$ then $a \otimes c$.
3. The relation $c \otimes c$ is false for at least one element c.
4. If $a \otimes b$ then there is an element x such that $a \otimes x$ and $x \otimes b$.
5. If S is an infinite sequence of elements a_k such that

$$a_k \otimes a_{k+1}, \quad a_k \otimes c \qquad (k = 1, 2, 3, \cdots)$$

(where c is some fixed element), then there is an element A having the following two properties: $a_k \otimes A$ whenever a_k belongs to S; and if $A' \otimes A$ then there is an element a_r of S such that $A' \otimes a_r$.

6. If a, b and $b \oplus a$ belong to the assemblage, then $a \oplus b = b \oplus a$.
7. If a, b, c, $a \oplus b$, $b \oplus c$ and $a \oplus (b \oplus c)$ belong to the assemblage, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
8. For every two elements a and b there is an element x such that $a \oplus x = b$.
9. If $x \otimes y$ then $a \oplus x \otimes a \oplus y$.

The second set contains these nine postulates and also the following four:

10. The product $a \odot b$ always belongs to the assemblage.
11. First distributive law: $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.
12. Second distributive law: $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.
13. If 0 is an element such that $c \oplus 0 = c$ for every element c, and if $0 \otimes a$ and $0 \otimes b$, than $0 \otimes a \odot b$.

2. The usual plan of defining the elementary functions first for real variables and afterwards, by a process of generalization, for complex variables, involves much unnecessary repetition in

the proofs of many of their properties. In Dr. Huntington's second paper the attempt is made to avoid this repetition by defining the functions once for all for the most general case. In the first place, the fundamental theorems concerning definite integrals along a path in the complex plane (especially Goursat's theorem that the integral of a differentiable function around a closed path is zero) are rigorously developed without presupposing any knowledge of functions of a real variable, except addition, subtraction, multiplication and division. Next, $\log z$ is defined as the integral of dz/z from 1 to z, and the fundamental relation $\log a + \log b = \log (ab)$ obtained by a familiar device.* Then e^z is defined as the inverse of $\log z$, and x^y as $e^{y \log x}$. The existence and properties of these functions follow immediately for the most general case, without separate consideration of irrational and imaginary values. Finally, the integration of such functions as $1/(1 + z^2)$ and $1/\sqrt{1 - z^2}$ leads very naturally to the definitions of $\tan^{-1} z$ and $\sin^{-1} z$ as abbreviations for certain logarithmic expressions, while $\tan z$ and $\sin z$ appear as the inverses of $\tan^{-1} z$ and $\sin^{-1} z$ respectively.

3. Dr. Keyser's paper undertakes a critical examination (1) of Dedekind's "theorem of complete induction," and (2) of the proofs offered by Dedekind and Bolzano of the so-called existence theorem for infinite assemblages. In respect to (1), it is shown that that theorem, so far from being "*the* scientific basis of the method of argument called complete induction," can itself be proved by means of mathematical induction provided one recognizes that our faith in the validity of, in the apodictic certainty yielded by, this argument form finds adequate justification in a certain a priori judgment (apparently first formulated as such by Poincaré), here called the axiom of infinity and shown to be a presupposition of all logical discourse. It follows that against all such "proofs" as are referred to under (2) there must lie the charge of circularity. In course of the article several theorems in the doctrine of Ketten are established.

4. The following extract indicates the nature of Professor Darwin's paper, which will appear in the *Transactions:*

*Cf. H. Burkhardt, Analytische Functionen, p. 162, and an article by J. W. Bradshaw about to appear in the *Annals of Mathematics.*

"Spherical harmonics render the approximate determination of the figure of a rotating mass of liquid a very simple problem. If ρ be the density, e the ellipticity, and ω the angular velocity of the spheroid, the solution is

$$\frac{\omega^2}{2\pi\rho} = \frac{8}{15}e.$$

This result is only correct as far as the first power of the ellipticity, but M. Poincaré has recently shown (*Philosophical Transactions*, volume 198 *A*, pages 333–373) how harmonic analysis may be used so as to give results as far as the squares of small quantities; and I have myself used his suggestion for the determination of the stability of the pear-shaped figure of equilibrium (*Philosophical Transactions*, volume in the press).

Both these papers involved the use of ellipsoidal harmonic analysis and it would be rather tiresome to extract the method from the complex analysis in which it is embedded. It therefore seems worth while to treat the well-worn subject of Maclaurin's spheroid as an example of the method in question."

5. Professor Hancock's paper is in summary as follows. In order that the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt,$$

where F is a regular one-valued function of its arguments, may have a maximum or minimum value, it is necessary that the function F have the two properties: 1° It must be homogeneous of the first degree in x' and y'; 2° The partial derivatives $\partial F/\partial x'$ and $\partial F/\partial y'$ must be continuous, even if there are sudden changes in x' and y'. We have further the four conditions: 1° The differential equation $G = 0$ must be satisfied for every point of the curve within the interval $t_0 \cdots t_1$; 2° The function F_1 must retain the same sign within the interval, and cannot become zero or infinite within this interval; 3° There cannot be conjugate points within this interval; 4° The function $E(x, y, p, q, \bar{p}, \bar{q})$ must always have the same sign within the interval $t_0 \cdots t_1$. The conditions 2° and 3° are necessary to establish the condition 4°.

8. The object of Professor Taylor's paper is, in extension of a problem started by Muir, to determine independently the co-

efficient of any alternant occurring in the product of a symmetric function of weight w and the alternant $|0123\cdots n-1|$. Proceeding from the standpoint of Muir, the symmetric function is regarded as being first expressed in the usual manner as a sum of products of elementary symmetric functions of weight w with certain coefficients. The problem then becomes, to find the coefficient of any alternant in the product of the single alternant and a product of elementary symmetric functions of weight w. A general reduction formula for such a coefficient is found. In particular the coefficient of the alternants of the type $|0123\cdots r_1 s_1 \cdots r_2 s_2 \cdots r_3 s_3|$, where $s_1 - r_1 = 2$, $s_2 - r_2 = 2$, $s_3 - r_3 = q$, in the product $|012\cdots n-1| \sum a_1 a_2 \cdots a_{n-i} (\sum a_1)^i$ is found to be

$$\frac{i(i-1)\cdots(i-s_1+2)}{\lfloor s_1 - 1} \cdot \frac{(s_1-2)(s_1-3)\cdots(s_1+r_3-s_3-2)}{\lfloor s_3 - r_3 - 1}.$$

If tables of coefficients for weight w be made, others of the same type for weight $w + j_h$ can be constructed by means of the preceding reduction formula. Tables of this kind from weight one to weight seven were given. The coefficient of any alternant in the product of the original symmetric function and $|0123\cdots(n-1)|$ can then be obtained by multiplying the coefficient of the table of weight w by the coefficient of the elementary symmetric function product in the symmetric function table, and by taking the sum of all such products.

9. In Professor Roe's paper, which is an extension of a previous one by the author, the direct product of the symmetric function $\sum' \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}$ and the alternant $|\lambda_1 \lambda_2 \cdots \lambda_m|$ is considered. The coefficient of any alternant $|\kappa_1 \kappa_2 \cdots \kappa_m|$ in the product is first shown to be equal to the coefficient of $\alpha_{p_1} \alpha_{p_2} \cdots \alpha_{p_m}$ in the determinant

$$\begin{vmatrix} a_{\kappa_1-\lambda_1} & a_{\kappa_2-\lambda_1} & \cdots & a_{\kappa_m-\lambda_1} \\ a_{\kappa_1-\lambda_2} & a_{\kappa_2-\lambda_2} & \cdots & a_{\kappa_m-\lambda_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\kappa_1-\lambda_m} & a_{\kappa_2-\lambda_m} & \cdots & a_{\kappa_m-\lambda_m} \end{vmatrix}$$

and then a general recurrence formula for the coefficient is established. A method of calculating it by means of bringing about the identity of two complexes is also explained. It is

next shown how a product table for all symmetric functions of the preceding type of weight w and the alternant $|012\cdots m-1|$ can be so constructed by placing conjugate partitions in a vertical, and conjugate alternants in a horizontal line, equidistant from each end, that all the coefficients lie in a triangle whose hypotenuse consists of ones, and the sum of whose columns is always zero, excepting the last one which is unity. For this table general formulas are given for the coefficients. Moreover it is shown that the whole table is absolutely invariant and still gives the complete products if $|012\cdots m+r-1|$ is used as multiplier. It is shown how the coefficients of a quotient table are certain determinants taken from the product table, and that Dr. Taylor's *product* table is the author's *quotient* table.

As it was the author's original object to obtain a method of computing symmetric functions, it is finally shown that the table is a table for all the symmetric functions of weight w written in the vertical line.

10. In his second paper Professor Roe stated for the quotient table of alternants mentioned in his first paper, eight properties which he has already observed.

11. In Professor Lovett's paper attention is called to a group of $(2n-1)(n-1)$ transformations, in as many variables, which the author encountered in a study of the problem of n bodies, the law of force being an arbitrary function of the distance. The invariants of this group are functions of the integrals of the problem of n bodies which are independent of the law of force. This interpretation appears by choosing as the variables certain quadratic functions of the coördinates and velocities of the bodies of the system.

The paper itself is concerned with the case where n is arbitrary, but the group for the particular case $n=4$ may be presented in this brief abstract as a typical one; its twenty-one infinitesimal transformations may be written as follows:

$$X_{1,123}f \equiv 2s_{1,q_1} + r_{1,p_1} + s_{21,q_{12}} + s_{31,q_{13}} + r_{21,p_{12}} + r_{31,p_{13}},$$

$$X_{2,123}f \equiv 2s_{1,r_1} + q_{1,p_1} + s_{12,r_{12}} + s_{13,r_{13}} + q_{21,p_{21}} + q_{31,p_{31}},$$

$$X_{3,123}f \equiv 2s_{21,r_1} + 2s_{12,r_2} + q_{12,q_1} + q_{12,q_2} + q_{1,r_{12}} + q_{2,r_{12}} + s_{13,r_{23}} + s_{23,r_{31}} + q_{1,p_{13}} + q_{2,p_{21}} + q_{33,p_{31}} + q_{31,p_{32}},$$

$$X_{4,123}f \equiv 2q_{1,q_1} - 2r_{1,r_1} + q_{12,q_{12}} + q_{13,q_{13}} - r_{12,r_{12}} - r_{13,r_{13}} + s_{12,s_{12}} + s_{13,s_{13}} - s_{21,s_{21}} - s_{31,s_{31}},$$

$$X_{5,123}f \equiv 2q_{12,q_1} - 2r_{12,r_2} + s_{21,s_1} - s_{21,s_2} + q_{2,q_{12}} + q_{32,q_{13}} - r_{1,r_{12}} - r_{31,r_{23}} + s_{2,s_{12}} - s_{1,s_{12}} + s_{23,s_{13}} - s_{31,s_{32}},$$

$$X_{6,123}f \equiv 2q_{12,q_2} - 2r_{12,r_1} + s_{12,s_2} - s_{12,s_1} + q_{1,q_{12}} + q_{31,q_{23}} - r_{2,r_{12}} - r_{32,r_{31}} + s_{1,s_{21}} - s_{2,s_{21}} + s_{13,s_{23}} - s_{32,s_{31}},$$

$$X_{7,123}f \equiv 2s_{12,q_1} + 2s_{21,q_2} + r_{12,s_1} + r_{12,s_2} + s_{1,q_{12}} + s_{2,q_{12}} + s_{31,q_{23}} + s_{32,q_{31}} + r_{2,s_{12}} + r_{1,s_{21}} + r_{23,s_{13}} + r_{31,s_{23}},$$

$$X_{i,231}f, \quad X_{i,312}f, \quad (i = 1, 2, 3, \cdots, 7),$$

where $a_{i,b_j} \equiv a_i \partial f/\partial b_j$; $q_{ij} = q_{ji}$; $r_{ij} = r_{ji}$; $s_{ij} \neq s_{ji}$.

12. Dr. Rabinovitch considered the surface of revolution $f(\sqrt{x^2+y^2}, z) = 0$, referred to rectangular coördinates, the axis of z being the axis of revolution. The meridian planes (1) $y = 0$ and (2) $y = mx$, will cut each parallel circle $z = k$, $x^2 + y^2 = \rho^2$ into four segments, supplementary two by two. Describing upon these in each parallel plane the lunes of Hippocrates, we obtain four solid lunes, whose volume is proved to be equal to the volume of the common portion of the two intersecting cylinders, whose directrices are the meridian curves of the given surface of revolution, situated respectively in the planes (1) and (2), and whose corresponding generatrices are respectively parallel to the intersections of each parallel plane $z = k$ with (2) and (1). In the particular case of surfaces of revolution of the second degree, the volume of these lunes, between two parallel planes, does not involve any other irrationality than that which may be involved in the arbitrary distance $z_1 - z_0$ between the bounding planes. Thus, the expression for the volume of a solid, bounded by portions of a surface of revolution is free from the irrationality π.

In particular, the total volume of the lunes upon a sphere of radius a equals

$$\tfrac{1}{2} \cdot 8 \sin\theta \int_0^a (a^2 - z^2) dz = \tfrac{8}{3} a^3 \sin\theta,$$

θ being the angle of inclination of the meridional planes (1) and (2). For an hyperboloid of revolution of one sheet, where

the meridian curve is an equilateral hyperbola, the volume of the lunes between the planes $z = z_0$ and $z = z_1$ equals

$$4 \sin\theta \int_{z_0}^{z_1} (a^2 + z^2)dz.$$

The analogy between these solid lunes and the circular lunes of Hippocrates is evident. All other ellipsoidal and hyperboloidal lunes are easily reduced to these. Paraboloidal lunes are also considered.

14. The connex (2, 2) of the ternary domain $a_x^2 u_a^2 = 0$ coordinates to a line conic u_γ^2, the line conic $a_\gamma^2 u_a^2$; to a point conic c_x^2, the point conic $c_a^2 a_x^2$. If point and line conics be considered as flat spaces and points respectively in an S_5, the above coördination is expressed analytically as a connex (1, 1) in S_5. To a property of the connex (1, 1) in S_5 invariant under the projective group G_{35} of S_5 will correspond a property of the connex (2, 2) in S_2 invariant under the group G_8 of the plane which appears in S_5 as a subgroup of G_{35}.

Hence the invariant theory of the ternary connex (2, 2) breaks into two parts, one part invariant only under the projective group of the plane, the other invariant under a larger group. This second part may be very advantageously treated by reference to an S_5. Clebsch's (Lindemann) treatment of the connex (1, 1) in S_2 may be followed out for the connex (1, 1) in S_5 and the results obtained readily translated to apply to the connex (2, 2) in S_2.

Some facts as to the first part also are readily proven by considering the projective group of the plane as a subgroup of G_{35} in S_5 which leaves invariant a certain 4-spread and 2-spread. *E. g.*, The necessary and sufficient condition that the two quartics $a_x^2 b_x^2 (a\beta y)^2$ and $u_a^2 u_\beta^2 \ (abv)^2$ may be put respectively in the forms

$$\sum_1^6 (c_x^2)^2 \quad \text{and} \quad \sum_1^6 (u_\gamma^2)^2,$$

where the six conics c_x^2 touch the line v and the six conics u_γ^2 go through the point y, is that $(abv)^2(a\beta y)^2 = 0$.

15. Veronese, Segre and others have considered what may be termed the geometry of conics in which the conic defined

by six homogeneous coördinates $r_{11}: r_{22}: r_{33}: r_{23}: r_{13}: r_{12}$ is regarded as the element. Dr. Kasner's first paper is a contribution to this aspect of geometry. A linear equation in the coördinates r_{ik} defines a linear system of conics whose theory is well known. The general quadratic equation defines the quadratic system Q which is studied in detail by the author. It consists of ∞^4 conics, two of which pass through an arbitrary set of four points. The discussion of the degenerate conics contained in the system leads first to a correspondence defined by a quadri-quadric form f in two sets of line coördinates, and second to a curve of fourth class f, which is the envelope of the double lines contained in Q. The quartic covariant curve f is entirely general; to any quartic there correspond ∞^5 quadratic systems Q. The conics common to Q and one, two, or three linear systems are discussed. Finally analogous results are obtained for a quadratic system of quadric surfaces.

17. Peano has shown that a continuous curve

$$x = f(t), \quad y = \phi(t), \quad 0 \leqq t \leqq 1,$$

may fill a two dimensional region, in particular, the interior of a square. Peano's curve, however, has multiple points. If such points are excluded, then a curve defined by the above equations is called a Jordan curve. The question presents itself: May the external area of a Jordan curve have a positive value? This question is answered in the affirmative in Professor Osgood's second paper by means of an example. The paper has been published in the January number of the *Transactions*.

18. Mr. Young's paper is a continuation of the paper presented by him to the Society at its last summer meeting. He considers the theta-fuchsian functions of Poincaré associated with the group Γ of character $\{0, 3; 2, 4, \infty\}$ (Fricke-Klein notation), which has for fundamental circle the real axis. Every theta-fuchsian function of Γ of order $4\kappa + \alpha$ which does not become infinite at any point in the positive half-plane is the product of κ such theta-fuchsian functions of order 4 and one of order α, and has just κ movable zeros in a fundamental region C of Γ, unless $\alpha = 1$; in the latter case, it is the product of $\kappa - 1$ functions of order 4 and two of orders 2 and 3 respectively, and has just $\kappa - 1$ movable zeros in C. Theta-

fuchsian functions of any order exist with the proper number of arbitrarily assigned movable zeros. It is shown, further, how every such function can be rationally expressed in terms of two functions θ and ϕ, which are simple rational functions of two particular hyperelliptic theta constants. The functions θ and ϕ each vanish in a single point of C, and are such that θ^2 and $\theta\phi$ are theta-fuchsian functions of Γ of orders 2 and 3 respectively.

20. Scheffers has classified all hypercomplex number systems into quaternion or non-quaternion groups. Dr. Hawkes's paper gave a method for enumerating all types of non-quaternion systems of order n from types of order $n-1$. These types follow the usual principles of classification in being inequivalent, irreducible, non-reciprocal, and with moduli. The method deduced is founded on certain theorems of Benjamin Peirce as extended by Dr. Hawkes in the *Transactions*, volume 3.

21. Dr. Stecker's paper justifies certain forms of equations of doubly infinite systems of geodesic lines used in a paper presented a year ago, where such systems were projected conformally into systems of conics.

22. The problem discussed in Dr. Kasner's second paper may be stated as follows: Given a doubly infinite system of plane curves $F(x, y, \lambda, \mu) = 0$, to investigate the existence and properties of those surfaces which can be represented point by point upon the plane in such a manner that the geodesics of the surface are represented by the assigned curves. Beltrami solved the problem for the straight lines, the corresponding surfaces being those of constant curvature. Dr. Stecker (volumes 2 and 3 of the *Transactions*) has considered the case of a linear system

$$\lambda\phi(x, y) + \mu\psi(x, y) + \chi(x, y) = 0,$$

imposing the additional requirement that the representation should be conformal. This case, however, may be reduced to Beltrami's; the surfaces are simply those of constant curvature, and the linear system, for conformal representation, must be conformally equivalent to a linear system of circles.

In the case of an arbitrary system $F = 0$ no solution in general exists. The necessary and sufficient conditions depend

upon the consistency of a set of four partial differential equations in three unknown functions. The discussion of the number of solutions is carried out by means of Dini's theorem, the surfaces of Liouville type playing an exceptional rôle. Among the important results obtained are the two following: The only surfaces whose geodesics are capable of representation by circles are those of constant curvature. The only surfaces whose geodesics can be conformally represented by curves of the form $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ are the developable surfaces and those whose linear element may be written $ds^2 = v(du^2 + dv^2)$. The paper will appear in the *Transactions*.

23. Many properties of Laplace's equation in two dimensions can be deduced from the theory of functions of a complex variable. In this way the two following theorems may be easily proved, in each of which we suppose the function $u(x, y)$ to be harmonic throughout the neighborhood of a point P: 1° If u becomes infinite for *no* method of approaching P, it will, if properly defined at P, be harmonic there. 2° If u becomes infinite for *every* method of approaching P, it has the form $C \log r + v$ where C is a constant, r the distance from P, and v a function which is harmonic at P. According to a remark of Professor Osgood, the first of these theorems is an immediate consequence of the second. In the present paper Professor Bôcher shows how the second theorem (and therefore also the first) can be extended to cases to which the original method of proof was in no wise applicable — on the one hand to Laplace's equation in three dimensions, and on the other hand to the general linear partial differential equation of the second order of the elliptic type in two independent variables and with analytic coefficients. The method of proof consists in an application of Green's theorem.

24. In the paper of Mr. Birkhoff and Mr. Vandiver the complement of two well-known arithmetical congruence theories is considered. The two theories (given in most text-books) are: 1° Theory of power residues; 2° Theory of the solution of binomial congruences. But the relation $A^n - 1 \equiv 0 \pmod{p}$, or more generally $a^n - b^n \equiv 0 \pmod{p}$, where A, a, b and p are integers, may be considered from another standpoint; given a, b and n, we may ask the question: What are the properties of the divisors of $a^n - b^n$, or will certain classes of

divisors always exist? The theory arising from efforts to solve these problems may be regarded as a complement to the two theories just mentioned, and forms the subject of the present discussion.

A primitive divisor of $a^n - b^n$ is defined as a divisor which is prime to all the factors of the form $V_k = a^k - b^k$, k being less than n. Then, using several theorems due to Euler and Sylvester (which are also proved), the following theorem (which we believe new) is arrived at: If $n > 2$, then every integral form $a^n - b^n$ possesses at least one primitive divisor, with the single exception $2^6 - 1$.

The paper concludes with applications of the theory of cyclotomic divisors to the "instantaneous" demonstration of the two theorems: 1° If n is any integer, there is an indefinite number of primes of the form $1 \pmod{n}$. 2° The equation $(x^p - 1)/(x - 1) = 0$ is irreducible if p is prime.

25. The main steps in the proof given in Professor Miller's paper are as follows: It is first observed that the continued product of all the operators of an abelian group G is the identity whenever G contains more than one operator of order 2. If G contains only one operator S of order two, then S is always the continued product of all the operators of G. The group of isomorphisms I of a cyclic group C must contain more than one operator of order 2 unless the order of C is one of the three numbers 4, p^α, $2p^\alpha$, where p represents any odd prime. Without employing the theory of congruences it is proved that in these three special cases I contains just one operator of order 2. The generalized Wilson's theorem follows directly from these facts. The paper has been offered to the *Annals of Mathematics* for publication.

26. In Professor Morley's paper the factors of the determinant $|(x_i - a_j)^{-2}|$ are found by a process which appears to be novel; and thereby the geometric theorem is proved that, given a rational norm curve R_n through $n + 1$ points in S_n, a quadric apolar to the $n + 1$ points cuts R_n in points which form a system conjugate with the $n + 1$ spaces S_{n-1} determined by the $n + 1$ points.

F. N. COLE.

COLUMBIA UNIVERSITY.

THE DECEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE second regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, December 20, 1902, at the University of California. The following sixteen members were present:

Professor R. E. Allardice, Dr. E. M. Blake, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor L. M. Hoskins, Dr. D. A. Lehmer, Dr. J. H. McDonald, Professor G. A. Miller, Dr. A. C. Moreno, Dr. C. A. Noble, Dr. T. M. Putnam, Professor Irving Stringham, Dr. S. D. Townley, Mr. A. W. Whitney, Professor E. J. Wilczynski.

A morning and an afternoon session were held, Professor Stringham acting as chairman at both sessions. The officers elected at the May meeting were reëlected for one year, and the by-law relating to the name of the section was amended by replacing "Pacific" by "San Francisco."

The following papers were read at this meeting:

(1) Professor R. E. ALLARDICE: "On a system of similar conics through three points and its transformation group."

(2) Professor H. F. BLICHFELDT: "On a property of conic sections."

(3) Professor L. E. DICKSON: "Generational relations for the abstract group G simply isomorphic with the linear fractional group in the Galois field of order p^n."

(4) Professor L. M. HOSKINS: "A simple method of determining the free nutation of a yielding spheroid."

(5) Dr. D. N. LEHMER: "On the parametric representation of the tetrahedroid surface."

(6) Professor A. O. LEUSCHNER: "Elimination of aberration and parallax in calculating preliminary orbits."

(7) Mr. W. A. MANNING: "The positive primitive substitution groups of class $2p$, p being any prime."

(8) Professor G. A. MILLER: "On the holomorph of a cyclic group."

(9) Dr. C. A. NOBLE: "A problem in relative minima."

(10) Dr. S. D. TOWNLEY: "The probability of collisions amongst the stars."

(11) Professor E. J. WILCZYNSKI: "On a certain congruence associated with a given ruled surface."

Professor Dickson's paper was read by the secretary. In the absence of Professor Leuschner his paper was read by title. All the other papers were read by their authors and most of them aroused considerable discussion. Professor Blichfeldt's paper appears in the present number of the BULLETIN. Abstracts of the other papers are given below.

1. On considering the loci and envelopes connected with a system of similar conics through three fixed points, Professor Allardice found that a number of curves occur which are intimately associated with the circle. In looking for an explanation of this fact, he found that such a system of conics possesses a group of transformations of the following character; Each transformation is equivalent to a quadric inversion ($x = 1/x$, etc.), followed by a rotation about the circumcenter, and then by a second quadric inversion. Any transformation of the group transforms any conic through the given points into a similar conic through the same points; but the transformations are themselves independent of the eccentricity of the conic. The path curves of the group were determined.

3. Professor Moore has given (see Dickson's Linear Groups, page 300) a set of generational relations for the group G considered by Professor Dickson. The object of the present paper is to give a simpler set of such relations, the proof being confined as yet to the twenty-two cases in which $p^n < 49$. Two cases $p = 2$ and $p > 2$ are essentially different.

If $n = 1$ and p is an odd prime, a complete set of generational relations for G is

$$T^2 = S^p = (ST)^3 = I, \quad (S^\tau T S^{2/\tau} T)^2 = I$$

where $\tau = 3, \cdots, \frac{1}{2}(p-1)$, but only one of the positive residues modulo p of τ, $2/\tau$, $-2/\tau$ is retained.

For $p = 3$ or 5, the only relations are

$$T^2 = I, \quad S^p = I, \quad (ST)^3 = I.$$

Additional relations are:

$$p = 7, \quad (S^3TS^3T)^2 = I.$$

$$p = 11, \quad (S^3TS^6T)^2 = I, \quad (S^4TS^6T)^2 = I.$$

$$p = 13, \quad (S^3TS^5T)^2 = I, \quad (S^4TS^7T)^2 = I.$$

$$p = 17, \quad (S^3TS^{12}T)^2 = I, \quad (S^4TS^9T)^2 = I,$$
$$(S^6TS^6T)^2 = I, \quad (S^7TS^{10}T)^2 = I.$$

$$p = 19, \quad (S^3TS^7T)^2 = I, \quad (S^4TS^{10}T)^2 = I,$$
$$(S^5TS^8T)^2 = I, \quad (S^6TS^{13}T)^2 = I.$$

$$p = 23, \quad (S^3TS^{16}T)^2 = I, \quad (S^4TS^{12}T)^2 = I,$$
$$(S^5TS^8T)^2 = I, \quad (S^6TS^8T)^2 = I, \quad (S^9TS^{13}T)^2 = I.$$

$$p = 29, \quad (S^3TS^{20}T)^2 = I, \quad (S^4TS^{15}T)^2 = I, \quad (S^5TS^{12}T)^2 = I,$$
$$(S^6TS^{10}T)^2 = I, \quad (S^7TS^{21}T)^2 = I, \quad (S^{11}TS^{16}T)^2 = I.$$

For any n and any odd prime p, the set is

$$T^2 = I, \quad (S_1T)^3 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda, \mu \text{ any marks}),$$
$$(S_\tau TS_{2/\tau}T)^2 = I \qquad (\tau \text{ any mark} \neq 0).$$

These relations are somewhat redundant. For $p^n = 9$, the group of order 360 is generated by T, S_1 and S_j, subject only to the relations

$$T^2 = S_1^2 = S_j^3 = I, \quad S_1S_j = S_jS_1, \quad (S_1T)^3 = I, \quad (S_jTS_jT)^2 = I,$$
$$(S_jS_1TS_j^{-1}S_1T)^2 = I.$$

Its isomorphism with the alternating group on six letters follows from the correspondence $T \sim (12)(34)$, $S_1 \sim (123)$ $S_j \sim (456)$.

For $p = 2$, the set of relations is

$$A^{2^n+1} = I, \quad B^2 = I, \quad (AB)^3 = I \quad (BA^rBA^s)^2 = I,$$

$r = 1, 2, \cdots, 2^n$, the value of S being uniquely determined modulo $2^n + 1$ by r. Among the pairs r, s always occur 1, 2;

2, 1; 2^n, $2^n - 1$; $2^n - 1$, 2^n; and the resulting relations follow from those giving the periods of A, B, and AB. For $n = 1$ or 2, there are no further relations. For $n = 3$, the relations are

$$A^9 = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (BA^3BA^3)^2 = I.$$

For $n = 4$, the relations are

$$A^{17} = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (BA^3BA^7)^2 = I,$$
$$(BA^4BA^{12})^2 = I, \quad (BA^6BA^9)^2 = I.$$

For $n = 5$, we may restrict r, s to the values 3, 25; 4, 10; 5, 17; 6, 15; 7, 11; 9, 14; 12, 20. The papers giving the proofs have been presented to the American and London Mathematical Societies.

4. The paper by Professor Hoskins discusses the free rotation of a nearly spherical body whose elastic properties and density have spherical symmetry about the center of mass. By taking as axes of reference the lines which at any instant would coincide with the principal axes of inertia if the strain due to rotation were annulled, the equations of angular momentum take simple forms. If A_0, B_0, C_0 are the principal moments of inertia for the unstrained body, and if I is the increment produced in the moment of inertia about the instantaneous axis by the centrifugal forces due to the actual angular velocity, the equations of motion are the same as for a rigid body whose principal moments of inertia are $A_0 + I$, $B_0 + I$ and $C_0 + I$. Since the differences of these quantities are the same as those of A_0, B_0 and C_0, and since I is a small fraction of each of the actual moments of inertia (though not necessarily of their differences), the period of the free nutation is to a close approximation the same as for a rigid body whose principal moments are A_0, B_0 and C_0. The result is general in that it is not restricted to the case in which the axis of rotation deviates but slightly from its mean position, nor to the case in which two of the principal moments of inertia are equal.

5. Dr. Lehmer employed the σ functions of Weierstrass in the parametric representation of the tetrahedroid surface and showed the arrangement of the singular points, the singular

planes, and their interrelations. The 120 lines joining the 16 singular points were shown to be the 120 lines in which the 16 angular planes intersect. Each of these lines meets 12 others in 6 points in involution. The paper has appeared in the January number (volume 25, number 1) of the *American Journal of Mathematics*.

6. Professor Leuschner's paper begins with a discussion of the expediency of taking into account the corrections for aberration and parallax. The author confirms his former conclusions (Beiträge zur Kometenbahnbestimmung) that in certain cases the resulting orbit will be considerably improved by their elimination. He then reviews the existing methods of accomplishing this elimination and points out that no method of eliminating parallax exists which is applicable in all cases, as even the "reduction to the locus fictus" fails for small geocentric latitudes. The author then proposes some very simple and general formulas for eliminating the parallax by means of corrections to the rectangular heliocentric coördinates of the sun. The corrections depend upon the parallax factors and naturally are independent of the geocentric distance of the body. The paper will be published in the *Bulletin of the Lick Observatory*.

7. Mr. Manning obtains the following results. All positive primitive groups of class $2p$, p a prime > 3, contain a primitive subgroup of degree $2p+1$. None exist unless $2p+1$ is a prime or a power of 3. For every prime number > 3 satisfying one or the other of these conditions there are just two groups, one of degree $2p+1$ and the other of degree $2p+2$. When $2p+1$ is a prime the group of that degree is semimetacyclic and the group of degree $2p+2$ is the modular group. When $2p+1$ is a power of 3 the first group is a subgroup of the holomorph of the abelian group of order 3^β and type $(1, 1, \cdots)$. The group of degree $2p+2 = 3^\beta + 1$ is the Mathieu group of order $(3^\beta - 1)\,3^\beta\,(3^\beta + 1)/2$. The paper will be offered to the *Transactions* for publication.

8. It is known that the holomorph k of a cyclic group G is a complete group and that its commutator subgroup is G whenever the order g of G is odd. When g is even and greater than 2 the commutator subgroup of K is the subgroup of G whose order is $g/2$, and K is never complete. The main object

of Professor Miller's paper is to determine additional useful properties of K. In particular the order of every operator of K is determined and also the degree of each substitution when K is represented as a transitive substitution group of degree g. It is pointed out that the properties of the group of isomorphisms of G furnish a very simple proof of Fermat's theorem. Some of the properties of the group of isomorphisms of K when g is even are developed. This paper will be offered to the *Transactions*.

9. On a former occasion (dissertation, Göttingen, 1901), Dr. Noble proved the existence of a function $y = y(x)$ which would render the integral

$$\int_{x_0}^{x_1} f(y, y' ; x)dx$$

a minimum, provided the integrand satisfied the conditions

$$|f(y, y' ; x)| \geqq k > 0, \qquad \left| f(y, y' ; x)\frac{dx}{dy} \right| \geqq k > 0$$

(k arbitrary constant).

The method employed in this existence proof was new—suggested by Hilbert in his Göttingen lectures—in that the minimizing curve was progressively constructed from cluster points, and finally identified with the solution of the Lagrange differential equation by means of the appropriate Weierstrass E-function. The object of the present paper is to extend this existence proof to the case of an integral

$$\int_{x_0}^{x_1} f(y, y', y'' ; x)dx,$$

under analogous restrictions for the integrand $f(y, y', y'' ; x)$. Such an integral arises in those problems of relative minima in which it is desired to minimize an integral

$$\int_{x_0}^{x_1} F(y, y', z, z' ; x)dx,$$

by functions $y = y(x)$, $z = z(x)$ which satisfy the conditioning differential equation $z = \phi(y, y' ; x)$. The method is the same

as in the existence proof for the simpler case cited above, the theory of the Weierstrass *E*-function, as developed by Zermelo, serving to identify the resulting minimal curve with the Lagrange solution.

10. A collision between two stars has often been advanced as an explanation of the phenomena of new stars. Dr. Townley has computed the probability of a collision, assuming a uniform distribution of stars in a finite universe of radius of 400 light years. The method adopted for the solution of the problem brings in two series each containing over two million terms. Only roughly approximate values of the sums of these series have thus far been obtained, but it is hoped to obtain more accurate values of them before long. The computation gives an exceedingly small fraction for the probability of a collision; so small indeed as to make the hypothesis of a collision between two stars no longer tenable as a rational explanation of the phenomena of new stars. No reasonable modification of the assumptions upon which the computations are based will produce a probability sufficiently great to explain the observed number of new stars.

11. In a paper recently published in the *Transactions*, Professor Wilczynski has studied a congruence made up of all of the generators of the first kind on the hyperboloids osculating a given ruled surface. This congruence, which there made its appearance in connection with the theory of covariants of a system of linear differential equations, is considered more fully in the present paper. It is found that its focal surfaces are the flecnode surfaces of the original surface. The theory of the developable surfaces contained in the congruence, and the theory of certain other characteristic families of ruled surfaces of the congruence are fully developed. Certain theorems, whose formulation in the previous paper was somewhat inadequate, are completed. Incidentally a simple interpretation is found for the canonical form previously adopted by the author for a system of differential equations of the kind here considered. The paper will be offered to the *Transactions* for publication.

G. A. MILLER,
Secretary of the Section.

STANFORD UNIVERSITY.

THE ABSTRACT GROUP G SIMPLY ISOMORPHIC WITH THE ALTERNATING GROUP ON SIX LETTERS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, December 29, 1902.)

1. A SLIGHT correction of a theorem due to De Séguier * leads to the result that G is generated by three operators a, b, c, subject only to the relations

$$(1) \qquad a^2 = I, \quad b^4 = I, \quad c^3 = I, \quad (ac)^3 = I,$$

$$(2) \qquad (ab^{-1}ab)^3 = I, \quad (ab^{-2}ab^2)^2 = I,$$

$$(3) \qquad (cb^{-1}ab)^2 = I, \quad (cb^{-2}ab^2)^2 = I.$$

But these generators are not independent, since

$$(4) \qquad a = cb^{-1}cbc.$$

A simple verification of (4) results from the correspondence

$$a \sim (12)(34), \quad b \sim (12)(3456), \quad c \sim (123)$$

between the generators of the simply isomorphic groups.

It is shown in this section that G is generated by the two operators b and c, subject to the complete set of generational relations

$$(5) \qquad b^4 = I, \quad c^3 = I, \quad (b^{-1}cbc^{-1})^2 = I, \quad (b^2c)^4 = I.$$

These relations follow from (1), (2), (3); for, by the above correspondence, $b^{-1}cbc^{-1} \sim (14)(23)$, $b^2c \sim (1235)(46)$.

If a be defined by (4), *relations* (1), (2), (3) *follow from* (5).

$$a^2 = cb^{-1}cbc^{-1}b^{-1}cbc = c(b^{-1}cbc^{-1})^2c^{-1} = I,$$

$$(ac)^3 = cb^{-1}c^3bc^{-1} = I.$$

**Journal de Math.*, 1902, p. 262. For $y=2, \ldots, n-3$ in his formula (6), should stand $y=1, \ldots, n-4$.

As an auxiliary result, we note that

$$c(bcb^{-1}cb) = (bcb^{-1}cb)c. \tag{6}$$

The condition (6) may be given the successive forms

$$cbcb^{-1}\cdot cbc^{-1}b^{-1}\cdot c^{-1}bc^{-1}b^{-1} = I,$$

$$cbcb^{-1}\cdot bcb^{-1}c^{-1}\ \ c^{-1}bc^{-1}b^{-1} = (cbc^{-1}b^{-1})^2 = I \quad [\text{by } (5_3)].$$

Since (6) may be written $cbac^{-1} = ba$, we have

$$c\cdot ba = ba\cdot c. \tag{7}$$

In view of (6) and (5_3), we get

$$(cb^{-1}cb)^3 = cb^{-1}\cdot bcb^{-1}cb\ \ c^2b^{-1}cb = I. \tag{8}$$

To verify (3_1) we note that, by (8),

$$cb^{-1}ab = cb^{-1}\cdot cb^{-1}cbc\cdot b = cb^{-2}c^{-1}bc^{-1}b^{-1}c^{-1}b^2.$$

In the indicated square of the latter, we replace b^2cb^{-2} by $c^{-1}b^2c^{-1}b^2c^{-1}$, in view of (5_4), and transform by cb^2, and get

$$\begin{aligned}
&c^{-1}\cdot bc^{-1}b^{-1}c\cdot b^2c^{-1}b^2\cdot cbc^{-1}b^{-1}\cdot c^{-1}b^2cb^2\\
&= c^{-1}\cdot c^{-1}bcb^{-1}\cdot b^2c^{-1}b^2\cdot bcb^{-1}c^{-1}\cdot c^{-1}b^2cb^2 \quad [\text{by } (5_3)]\\
&= cb\cdot cbc^{-1}b^{-1}\cdot cb^{-1}cb^2cb^2\\
&= cb\cdot bcb^{-1}c^{-1}\cdot cb^{-1}cb^2cb^2 = (cb^2)^4 = I \quad [\text{by } (5_4)].
\end{aligned}$$

To verify (3_2), we transform by b^2c^{-1} and get

$$\begin{aligned}
&cb^2cb^2\cdot cb^{-1}cb\cdot cb^2cb^2\cdot cb^{-1}cb\\
&= b^2c^{-1}b^2c^{-1}\cdot cb^{-1}cb\ \ b^2c^{-1}b^2c^{-1}\cdot cb^{-1}cb\\
&= b^{-1}(b^{-1}c^{-1}bc)^2b = I.
\end{aligned}$$

To verify (2_1), we note that

$$\begin{aligned}
ab^{-1}ab &= cb^{-1}cb\cdot cb^{-1}cb^{-1}cbcb\\
&= cb^{-1}cb(cb^{-1}cb^{-1}cbcb)^{-1} = cb^{-2}c^{-1}bc^{-1}bc^{-1}.
\end{aligned}$$

Cubing the inverse and transforming by c, we get (8).

To verify (2_2), we note that

$$\begin{aligned}
ab^{-2}ab^2 &= ac^{-1}\cdot cb^{-2}ab^2 = ac^{-1}\cdot b^{-2}a^{-1}b^2c^{-1}\\
&= cb^{-1}cb^{-1}c^{-1}b^{-1}c^{-1}bc^{-1}b^2c^{-1}.
\end{aligned}$$

Transforming its square by cb^2, we get

$$bcb^{-1}c^{-1}\cdot b^{-1}c^{-1}bc^{-1}\cdot bcb^{-1}c^{-1}\cdot b^{-1}c^{-1}bc^{-1}$$
$$= cbc^{-1}b^{-1}\cdot b^{-1}c^{-1}bc^{-1}\cdot cbc^{-1}b^{-1}\cdot b^{-1}c^{-1}bc^{-1}$$
$$= cbc^{-1}b^{-2}c^{-1}b^{2}c^{-1}b^{-2}c^{-1}bc^{-1}.$$

Transforming by cb and taking the inverse, we get $(b^2c)^4 = I$.

2. In a paper entitled "The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field," communicated November 2, 1902 to the London Mathematical Society, the writer shows that the group G is generated by three operators subject to the relations

$$(9) \qquad T^2 = I, \quad S_1^3 = I, \quad S_j^3 = I, \quad S_1S_j = S_jS_1,$$

$$(10) \qquad (S_1T)^3 = I, \quad (S_jT)^4 = I, \quad (S_jS_1TS_j^{-1}S_1T)^2 = I.$$

From these we obtain relations (1), (2), (3), if we set

$$a = T, \quad b = S_jT, \quad c = S_1.$$

This is evident for relations (1). Also,

$$(ab^{-1}ab)^3 = (S_j^{-1}TS_jT)^3 = S_j^{-1}\cdot TS_jTS_j \;\; S_jTS_jT\cdot S_j^{-1}TS_jT$$
$$= S_j^{-1}\cdot S_j^{-1}TS_j^{-1}T\cdot TS_j^{-1}TS_j^{-1}T\cdot S_j^{-1}TS_jT = (S_jT)^4 = I.$$

Also (2_2) and (3_2) follow from (9) and $(S_jT)^4 = I$, while (3_1) follows from (9) and the first and third relations (10). We thus obtain a new proof that (9) and (10) define G.

We may readily derive directly from (9) and (10) a complete set of relations between the two generators $b = S_jT$ and $c = S_1$ of G. We note that, from (10_1),

$$T = S_1TS_1TS_1 = S_1\cdot TS_j^{-1}\cdot S_1\cdot S_jTS_1 = cb^{-1}cbc.$$

We therefore have

$$S_1 = c, \quad T = cb^{-1}cbc, \quad S_j = bc^{-1}b^{-1}c^{-1}bc^{-1}.$$

Then $(S_jT)^4 = I$ follows from (5_1), $S_1^3 = I$ from (5_2), $T^2 = I$ from (5_3), $S_1S_j = S_jS_1$ from (5_3). Thus

$$S_1S_j = cbc^{-1}b^{-1}\cdot c^{-1}bc^{-1} = bcb^{-1}c^{-1}\cdot c^{-1}bc^{-1}$$
$$= bc\cdot b^{-1}cbc^{-1} = bc\cdot cb^{-1}c^{-1}b = S_jS_1.$$

Since $S_1T = c^{-1}b^{-1}cbc$ is the transform of c by bc, it is of period three.

The final relation (10) becomes

$$(bc^{-1}b^{-1}c \cdot b^{-1}cbc)^2 = (c^{-1}bcb^{-1} \cdot b^{-1}cbc)^2 = (c^{-1}bcb^2cbc)^2$$
$$= c^{-1}b(cb^2)^4b^{-1}c = I.$$

Since S_j is commutative with S_1, the condition $S_j^3 = I$ follows from $(b^{-1}c^{-1}b^2c^{-1})^3 = I$ or $(cb^2cb)^3 = I$.

THE UNIVERSITY OF CHICAGO,
December 11, 1902.

NOTE ON A PROPERTY OF THE CONIC SECTIONS.

BY PROFESSOR H. F. BLICHFELDT.

(Read before the San Francisco Section of the American Mathematical Society, December 20, 1902.)

IT is easily proved that if P, Q, R are any three points on the conic $Ax^2 + By^2 = 1$, and O the center of the conic, then the areas of the triangles OPQ, OPR, OQR will satisfy an equation independent of the position of the points P, Q, R. If a, b, c are the areas in question, this equation is

$$(1) \quad a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.$$

Now we can prove that such an invariant relation is possible for no plane curves except the central conics; *i. e., if we seek a plane curve C and a point O in its plane such that, if P, Q, R are any three points on C, the triangles OQR, ORP, OPQ are connected by a relation independent of the coördinates of the points P, Q, R, we find C to be a central conic section and O its center.*

To prove this theorem, let O be the origin of coördinates, and let the coördinates of P, Q, R be respectively x_1, y_1; x_2, y_2; x_3, y_3. Then twice the areas of the three triangles are

$$2a = \pm (y_2x_3 - y_3x_2), \quad 2b = \pm (y_3x_1 - y_1x_3),$$
$$2c = \pm (y_1x_2 - y_2x_1),$$

which expressions are functions of the three independent variables x_1, x_2, x_3; y being considered a given function of x for points on the curve.

As a, b, c must satisfy a relation independent of x_1, x_2, x_3, the Jacobian $\partial(a, b, c)/\partial(x_1, x_2, x_3)$ must vanish. If y_1' represents dy_1/dx_1, etc., we find

$$y_3'\{y_3[x_1y_2 - x_2y_1 + x_1x_2(y_2' - y_1')] + x_3(x_2y_1y_1' - x_1y_2y_2')\}$$
$$+ x_3[(x_1y_2 - x_2y_1)y_1'y_2' + y_1y_2(y_2' - y_1')] + y_3(x_2y_1y_1' - x_1y_2y_2') = 0,$$

say

$$y_3'(y_3k + x_3l) + x_3m + y_3l = 0, \tag{2}$$

k, l, m being functions of x_1, x_2 only, and therefore independent of x_3.

Two cases (α) and (β) may now present themselves as follows:

(α) The functions k, l, m are not all identically zero. In this case the equation (2) gives, when integrated,

$$y_3^2k + 2y_3x_3l + x_3^2m = f(x_1, x_2). \tag{3}$$

If we give to x_1 and x_2 arbitrary constant values, the equation (3) represents a conic section with its center at O.

(β) The functions k, l, m, are all zero. We must then have $x_2y_1' - y_2 = 0$. Giving to y_1' a definite constant value, we obtain the equation of a straight line — a special case of (3).

The theorem stated above is therefore proved.

It may be noticed that $f(x_1, x_2)$ in (3) may be multiple valued. The equation will then represent a series of similar conics similarly placed. If these are finite in number, say n, we find that, if P, Q, R be located anywhere on this system of curves, the areas a, b, c of the three triangles considered will satisfy an equation of degree

$$6n + 18n(n - 1) + 6n(n - 1)(n - 2)$$

at most, whose left-hand member is composed of factors of form similar to (1), as the reader may prove without much difficulty.

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THE ANALYTIC THEORY OF DISPLACEMENTS.

BY R. W. H. T. HUDSON, M.A.

Introduction.

THE subject of displacements is connected with several branches of pure mathematics and can be approached from many sides. The most comprehensive point of view is, no doubt, to regard the subject as an illustration of the theory of groups, the group here involved being a particular case of the group of linear homogeneous transformations. Again, instead of the symbol of an infinitesimal transformation, we may use the matrix of the corresponding linear substitution, and by means of elementary properties of matrices much of the work may be shortened and simplified.

By a generalization of the ordinary notion, a displacement may be defined as a projective point transformation which does not alter distance as measured with reference to an absolute quadric locus, and which further is capable of being represented as the result of an infinite number of infinitesimal transformations of the same kind. The analytic representation of a displacement must be a linear transformation of the variables which leaves the absolute unaltered. Since by a suitable choice of real or imaginary coördinates the equation of the absolute may be expressed by means of the sum of the squares of the coördinates, the problem in its simplest algebraic form is reduced to finding the most general orthogonal transformation.

The main object held in view throughout this paper is to obtain the general parametric representation of a displacement in two or three dimensions on the basis of the definition given above, and the method adopted is to integrate the equations of infinitesimal transformation. The formulæ are developed in a natural manner from the beginning and in addition to elementary processes use is made of matrices and quaternions. Not much is said about the other kind of transformation, corresponding to an improper orthogonal matrix, that is, one whose determinant is negative, because the general transformation of this kind can be obtained by combining a suitable displacement with any particular reflexion, and accordingly the analytic

representation requires no further development; again since an improper orthogonal matrix can be converted into a proper one by adding another row and column, so a transformation which preserves distance but changes order can be regarded as a displacement in a space one dimension higher.

From the limited nature of the subject it is not to be expected that any really new theorem remains to be discovered, but the method of obtaining the equations of finite displacement, though a perfectly obvious one to those who are accustomed to work with matrices, does not seem hitherto to have been successfully applied to this particular purpose.* It is hoped that the formulæ will enable the reader to obtain a clearer grasp and easy proofs of the many interesting geometric theorems given in the text-books;† and also to illustrate that aspect of the subject which is expounded in treatises on continuous groups.‡

For convenience of manipulation the subject takes the form of the theory of screws in elliptic space,§ and may form an introduction to non-euclidean kinematics. This, however, is not essential and the analysis is intended to cover all systems of measurement with the help of known methods of extension and transition.

I. *Displacements in Two Dimensions.*

1. In space of two dimensions the absolute is taken to be

$$x_1^2 + x_2^2 + x_3^2 = 0;$$

and since the coördinates are homogeneous we may suppose that for points not on this locus

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

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*Cf. Beez, *Schlömilch's Zeitschrift*, vol. 43, pp. 65, 121, 227. Stringham, *Compte Rendu du 2^me Congrès International de Mathématiciens tenu à Paris*, 1900, p. 327. Cole, "Rotations in space of four dimensions," *Amer. Journ.*, vol. 12, p. 191.

†*E. g.*, Schönflies, "Geometrie der Bewegung"; see also Study, *Math. Annalen*, vol 39, p. 441.

‡*E. g.*, Lie, Transformationsgruppen, III, p. 198; see also *Proc. L. M. S.* Burnside, vol. 26, p. 33, and Baker, vol. 34, pp. 91 and 347; these two papers appeared after the above was written.

§See Buchheim, *Proc. L. M. S.*, vol. 15, p. 83, where Grassmann's methods are used, and vol. 16, p. 15; vol. 17, p. 240.

The general linear transformation is represented in matrix notation by

$$x' = Ax = xA',$$

where A is a three-rowed matrix and A' is its conjugate. The condition

$$x_1'^2 + x_2'^2 + x_3'^2 = x_1^2 + x_2^2 + x_3^2$$

gives

$$xA'Ax = xEx,$$

where E is the unit matrix, and since this is to hold for all values of x,

$$A'A = E;$$

and accordingly A is an *orthogonal* matrix.* Since $|A|^2 = 1$, distinct cases arise when $|A|$ is $+1$ or -1. Since the matrix of an infinitesimal transformation differs only slightly from E it must belong to the former class, *i. e.*, be *proper*. We shall find that the most general matrix of this class can be obtained by integrating the equations of infinitesimal transformation, and the finite transformation so obtained will be called a *displacement*.

Putting $A = E + P\delta t$ and neglecting δt^2 the condition for an orthogonal matrix is

$$(E + P'\delta t)(E + P\delta t) = E,$$

giving

$$P' + P = 0,$$

so that P is a *skew* matrix. The corresponding infinitesimal transformation is

$$x' = x + Px\delta t,$$

which may be written in the form of a differential equation

$$\frac{dx}{dt} = Px.$$

Integrating this on the hypothesis that P is independent of t,

* For a somewhat similar development of the subject see Study, *Mittheilungen aus dem naturwissensch. Verein für Neu-Vorpommern und Rügen in Greifswald*, 1899, p. 1.

and taking x for initial and x' for final values, we find that the finite transformation is*

$$x' = e^{tP}x.$$

To obtain the actual formulæ we must expand e^{tP} in powers of P and reduce to polynomial form by means of the equation satisfied by P.

Let

$$P = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

and

$$s^2 = a^2 + b^2 + c^2.$$

Then the matrix P satisfies the cubic equation

$$\begin{vmatrix} -P & -c & b \\ c & -P & -a \\ -b & a & -P \end{vmatrix} = 0$$

or

$$P^3 + s^2 P = 0,$$

whence

$$e^{tP} = E + tP + \frac{1}{2}t^2P^2 - \frac{1}{3!}t^3s^2P - \frac{1}{4!}t^4s^2P^2 + \cdots$$

$$= E + s^{-1}\sin st\, P + s^{-2}(1 - \cos st)\, P^2.$$

Since this matrix contains three independent parameters $a : b : c$ and t it must be the most general matrix of the class considered, for the nine elements of an orthogonal matrix are subjected to six relations.

It is evident that the point given by $Px = 0$ is unaltered by the transformation, which may therefore be called a *rotation* about the point $a : b : c$. Let r be the *distance* between any point x and the centre of rotation $a : b : c$; then by the ordinary formula of elliptic geometry

$$\cos r = s^{-1}(ax_1 + bx_2 + cx_3),$$

* By repeated differentiation $d^nx/dt^n = P^nx$, and the integral is simply Taylor's expansion. The expression of every real proper orthogonal substitution in this form is due to Taber, *Bull. of the N. Y. Math. Soc.*, vol. 3, p. 251.

and the distance between x and $x + \delta t Px$ is

$$\delta t\{(bx_3 - cx_2)^2 + (cx_1 - ax_3)^2 + (ax_2 - bx_1)^2\}^{\frac{1}{2}} = \delta t \cdot s \sin r.$$

Hence the *angle* of rotation is st, $= \theta$, say. We may now for convenience put $s = 1$ and then, since

$$E + P^2 = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix},$$

the first equation included in $x' = e^{\theta P} x$ is

$$x_1' = x_1 \cos\theta + \sin\theta\,(-cx_2 + bx_3) + (1 - \cos\theta)\,a(ax_1 + bx_2 + cx_3),$$

agreeing with the known formula for the rotation of a point in three-dimensional euclidean space about the direction $a : b : c$.*

2. We have thus obtained the general displacement in terms of three parameters; it remains to be shown how Euler's parameters arise in connection with Cayley's expression for an orthogonal matrix.

The matrix of the finite transformation is

$$e^{\theta P} = E + \sin\theta P + (1 - \cos\theta)\,P^2$$

$$= E + 2\cos^2\frac{\theta}{2}\left(\tan\frac{\theta}{2}P + \tan^2\frac{\theta}{2}P^2\right).$$

Put $\alpha = a\tan\dfrac{\theta}{2}$, $\beta = b\tan\dfrac{\theta}{2}$, $\gamma = c\tan\dfrac{\theta}{2}$,

$$\Pi = \tan\frac{\theta}{2}P = \begin{vmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{vmatrix}.$$

* For other methods of obtaining this formula see *Messenger of Math.*, Bromwich, vol. 29, p. 41, and Hudson, vol. 31, p. 151.

Then
$$\Pi^3 + \tan^2\frac{\theta}{2}\Pi = 0,$$
$$\tan^2\frac{\theta}{2}(E - \Pi) + E - \Pi^3 = \sec^2\frac{\theta}{2}E,$$
$$(E - \Pi)\left(\sec^2\frac{\theta}{2}E + \Pi + \Pi^2\right) = \sec^2\frac{\theta}{2}E.$$
$$\therefore e^{\theta P} = -E + 2(E - \Pi)^{-1}$$
$$= (E + \Pi)(E - \Pi)^{-1},$$

which is Cayley's form; the elements of Π are the parameters of the displacement. This expression is not valid when $\theta = \pi$; the matrix of the displacement then is $E + 2P^2$, and represents a half-turn about, or a reflexion in, the point a, b, c.

Conversely, when the orthogonal matrix A of the displacement is given, and $|A + E| \neq 0$, the parameters are the elements of the skew matrix $(A + E)^{-1}(A - E)$ and from them the centre and angle of rotation can be found.

3. The preceding analysis applies equally well to the displacement of a point in a plane in which the system of measurement is elliptic and to the displacement of a point on a sphere in ordinary space, and thence by a limiting process, to displacements in a euclidean plane. This process consists in putting $x_3 = 1$ and neglecting squares and products of the other coördinates. Taking
$$P = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -a \\ -b & a & 0 \end{bmatrix}, \quad E + P^2 = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 1 \end{bmatrix},$$

we easily find that the formulæ reduce to

$$x_1' = a + \cos\theta(x_1 - a) - \sin\theta(x_2 - b),$$
$$x_2' = b + \sin\theta(x_1 - a) + \cos\theta(x_2 - b),$$
$$x_3' = 1 = x_3,$$

which are the ordinary formulæ for rotation about the point a, b.

Since every improper orthogonal three-rowed matrix, for which the determinant is -1, is the product of a proper mat-

rix and the matrix $-E$, we have to consider the meaning of the transformation $x_1' = -x_1$, $x_2' = -x_2$, $x_3' = -x_3$. In the elliptic plane the points x and $-x$ are not distinct, or only nominally so, and all the transformations come under the head of displacements.* On the euclidean sphere the points x and $-x$ are diametrically opposite and the transformation $x' = -x$ is a *reflexion* in the centre. This combined with any rotation gives the most general transformation of the second kind.

In the euclidean plane the transformation $x' = -x$ is excluded by the hypothesis $x_3 = 1$, but every improper orthogonal matrix can be derived from a proper matrix by multiplying it by the improper matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which corresponds to a *reflexion* in the line $x_1 = 0$.

It is beyond the scope of this paper to develop the many interesting geometric theorems which can be proved very easily from these formulæ; we are concerned rather with the various methods of parametric representation.

4. The use of quaternions as a convenient method of formulating two-dimensional displacements may now be explained, for it follows easily from Cayley's expression for an orthogonal matrix. Since

$$(1 + i\alpha + j\beta + k\gamma)(iz_1 + jz_2 + kz_3) = -(\alpha z_1 + \beta z_2 + \gamma z_3) + i(z_1 - \gamma z_2 + \beta z_3) + j(z_2 - \alpha z_3 + \gamma z_1) + k(z_3 - \beta z_1 + \alpha z_2),$$

the components of the vector part are the quantities $(E + \Pi)z$; thus, if we write

$$q = 1 + i\alpha + j\beta + k\gamma, \qquad \zeta = iz_1 + jz_2 + kz_3,$$

$V q\zeta$ is equivalent to $(E + \Pi)z$, and similarly $V\zeta q$ is equivalent to $(E - \Pi)z$. Now the general orthogonal transformation

$$x' = (E + \Pi)(E - \Pi)^{-1}x$$

* For a discussion of this peculiarity of the elliptic plane see Klein, Nicht-Euk'idische Geometrie, I, p. 103.

can be divided into two parts

$$x' = (E + \Pi)z, \qquad x = (E - \Pi)z,$$

which, when expressed in quaternion notation, are

$$\xi' = Vq\zeta = q\zeta - Sq\zeta, \qquad \xi = V\zeta q = \zeta q - Sq\zeta,$$

from which, on eliminating ζ,

$$\xi' = q\xi q^{-1}.$$

We notice that q may be multiplied by any scalar without affecting the formula; one of the most convenient forms to take is

$$\alpha = \cos\frac{\theta}{2} q = \cos\frac{\theta}{2} + (ia + jb + kc)\sin\frac{\theta}{2},$$

$$= \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3,$$

say, so that $T\alpha = 1$ and then

$$\alpha^{-1} = \cos\frac{\theta}{2} - (ia + jb + kc)\sin\frac{\theta}{2}$$

$$= \alpha_0 - i\alpha_1 - j\alpha_2 - k\alpha_3.$$

The composition of rotations is effected very simply when the quaternion notation is used; for the transformation $x'' = \beta x'\beta^{-1}$ following $x' = \alpha x\alpha^{-1}$ leads to $x'' = \beta\alpha x\alpha^{-1}\beta^{-1} = \gamma x\gamma^{-1}$, where $\gamma = \beta\alpha$.

The geometric interpretation of this is the well-known theorem that successive rotations $2\pi - 2A$, $2\pi - 2B$ about the corners A, B of a spherical triangle are equivalent to a rotation $2C - 2\pi$ about the corner C.

5. One other important representation of rotations, namely as a linear transformation of the parameters of the generators of a unit sphere, is easily deduced from the quaternion formula

$$\xi' = q\xi q^{-1}$$

as follows.* We have at once $\xi' q = q\xi$. Now, on replacing k by ij or $-ji$,

*See Klein, Lectures on the Ikosahedron, Chap. II, § 2, and Cayley, *Math. Annalen*, vol. 15 (1879), p. 238. That a quaternion is equivalent to a two-rowed matrix was recognized by Sylvester, *Phil. Mag.*, vol. 16 (1883), p. 394.

$$\xi = ix_1 + (x_2 + ix_3)j, \qquad q = 1 + i\alpha + j(\beta - i\gamma).$$

After multiplying out and replacing j^2 by -1, we may equate coefficients of j and obtain

$$\begin{aligned} x_1'(i-\alpha) - (x_2' + ix_3')(\beta - i\gamma) &= (i-\alpha)x_1 - (\beta + i\gamma)(x_2 - ix_3) \\ - ix_1'(\beta - i\gamma) + (x_2' - ix_3')(1 + i\alpha) & \\ &= (\beta - i\gamma)ix_1 + (1 - i\alpha)(x_2 - ix_3), \end{aligned}$$

where now i may be regarded as equivalent to $\sqrt{-1}$; these may be altered to

$$\begin{aligned} (1 + x_1')(i-\alpha) - (x_2' + ix_3')(\beta - i\gamma) & \\ &= (i-\alpha)(1 + x_1) - (\beta + i\gamma)(x_2 - ix_3), \\ (1 - x_1')(\beta - i\gamma) - (x_2' - ix_3')(i-\alpha) & \\ &= (\beta - i\gamma)(1 + x_1) - (i + \alpha)(x_2 - ix_3), \end{aligned}$$

whence by division

$$-\frac{x_2' - ix_3'}{1 + x_1'} = \frac{(i+\alpha)(x_2 - ix_3) + (\beta - i\gamma)(1 + x_1)}{(\beta + i\gamma)(x_2 - ix_3) - (i-\alpha)(1 + x_1)},$$

which is the linear transformation of $(x_2 - ix_3)/(1 + x_1)$ referred to.

II. *Infinitesimal Displacements in Three Dimensions.*

6. Let A be the symmetric matrix associated with the absolute, so that its equation is

$$xAx = 0.$$

For convenience we shall suppose that $|A| = 1$ and that the actual values of the coördinates of any point not on the absolute are so chosen that $xAx = 1$.

The transformation $x' = Bx$ leaves the absolute unchanged if, for all values of x,

$$x'Ax' = xAx;$$

this requires

$$B'AB = A,$$

where B' is the matrix conjugate to B. For a small displacement, B differs slightly from the unit matrix, so let $B = E + \epsilon C$, where ϵ^2 may be neglected; then

$$C'A + AC = 0,$$

showing that AC is a skew matrix P, say, and then $C = A^{-1}P$ and the general infinitesimal displacement is

$$x' = x + \epsilon A^{-1}Px$$

where P is any skew matrix.

In this displacement the point x moves a small distance in the direction of the point $A^{-1}Px$. Now with any skew matrix P may be associated a certain screw, or linear complex, such that Px is the null plane of x. Hence $A^{-1}Px$ is the absolute pole of the null plane of x.

The polar screw Q is obtained by reciprocating with respect to the absolute. Thus $A^{-1}Px$ is the null point of the plane Ax with reference to Q, whence

$$A^{-1}P = Q^{-1}A,$$

so that the motion of x may be described as being towards the null point with reference to the polar screw of its absolute polar plane

The motion of any plane l is given by

$$l'x' = lx, \qquad x' = (E + \epsilon A^{-1}P)x;$$

whence

$$l' = l(E + \epsilon A^{-1}P)^{-1} = l(E - \epsilon A^{-1}P) = l + \epsilon PA^{-1}l,$$

on forming the conjugate matrix, so that the plane l turns about its intersection with $PA^{-1}l$, which is the null plane of the pole of l, and is the same as $AQ^{-1}l$ which is the polar of the null point of l with reference to the polar screw.*

7. The transformation

$$y = A^{-1}Qx, \qquad y' = A^{-1}Qx'$$

transforms the line xx' into the polar line of the intersection of the null planes $u = Qx$, $u' = Qx'$. If xx' is a ray of the complex Q, *i. e.*, if $x'Qx = 0$ then uu' is the same line; so this transformation has the property of transforming every line of a certain complex into its polar line. Consequently if a ruled surface is generated by lines belonging to the screw Q the linear point transformation whose matrix is $A^{-1}Q$ transforms the surface into its reciprocal with respect to the quadric A. †

* These theorems are given by Buchheim, *Proc. L. M. S.*, vol. 16, p.15.

†This is a more general statement of a theorem contained in a note in the *Messenger of Math.*, vol. 29, p. 191.

If we are reciprocating with respect to the surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

we have $A = E$ and the transformation is

$$y = Qx.$$

8. We shall consider next in some detail the case when $|P| = 0$ and the screw associated with the motion degenerates into a line. We cannot immediately apply the formulas for polar screws because the inverse matrices do not now exist. It is convenient to use two matrices in connection with a line. Let *

$$P = \begin{vmatrix} 0 & -r & +q & p' \\ +r & 0 & -p & q' \\ -q & +p & 0 & r' \\ -p' & -q' & -r' & 0 \end{vmatrix} \text{ and } P' = \begin{vmatrix} 0 & -r' & +q' & p \\ +r' & 0 & -p' & q \\ -q' & +p' & 0 & r \\ -p & -q & -r & 0 \end{vmatrix},$$

where p, q, r, p', q', r' are the coördinates of the line; then

$$PP' \equiv -(pp' + qq' + rr')\, E = 0.$$

Hence if Q, Q' are the matrices associated with another line, the two lines intersect if

$$PQ' + QP' = 0.$$

The lines are polar if

$$Q = AP'A, \qquad Q' = A^{-1}PA^{-1} \qquad (\text{since } |A| = 1),$$

for then the pole of any plane through one line lies on the other.

Hence the line P touches the absolute if it cuts Q, *i. e.*, if

$$PA^{-1}PA^{-1} + AP'AP' = 0.$$

For any other line we shall take this expression to be equal to -1.

* This must not be confused with Frobenius's notation for conjugate matrices; in the case of a skew matrix P the conjugate is simply P.

To find the distance from a point x to a line P we notice that the plane Px cuts the polar line Q in the point $Q'Px$ and the distance between this and x is

$$\cos^{-1}\frac{xAQ'Px}{\sqrt{xPQ'AQ'Px}}$$

Now

$$xPA^{-1}(PA^{-1}PA^{-1})Px = -xPA^{-1}Px - xPA^{-1}(AP'AP')Px$$

$$= -xPA^{-1}Px;$$

hence the distance from x to P is

$$\rho = \sin^{-1}\sqrt{-xPA^{-1}Px}.$$

The distance between x and its displaced position given by the transformation

$$x' = x + \epsilon A^{-1}Px$$

is

$$\sqrt{(\epsilon A^{-1}Px)A(\epsilon A^{-1}Px)} = \epsilon\sqrt{-xPA^{-1}Px} = \epsilon\sin\rho;$$

hence ϵ is the small angle of rotation about P. As in the case of two dimensions, so also here the general finite displacement is obtained by integrating

$$\frac{dx}{d\theta} = A^{-1}Px.$$

9. Returning to the general screw motion, let P be the screw and L any line; then it is possible to choose k so that $P - kL$ is a line, $= k'M$ say. Thus,

$$\epsilon A^{-1}P = \epsilon kA^{-1}L + \epsilon k'A^{-1}M$$

and the motion consists of two small rotations ϵk about L and $\epsilon k'$ about M. The most important case is when L and M are polar lines; they are then the *axes* of the screw and are of the form $P + \lambda Q$ where $|P + \lambda Q| = 0$. Discussion of this and other methods of resolution is simplified by taking the absolute in the form used in elliptic space. This will be done in connection with finite displacements.

III. *Finite Displacements in Three Dimensions.**

10. We have seen that the general infinitesimal displacement is

$$x' = x + \epsilon A^{-1} P x,$$

where ϵ is a small constant, A is the matrix of the absolute, and P is any four-rowed skew matrix. In what follows we shall take the absolute to be

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

so that $A = E$. By a suitable change of coördinates the absolute can be reduced to this form, and by admitting the possibility of imaginary coördinates we are able to include all kinds of space in one investigation.

The above transformation can be written as a differential equation

$$\frac{dx}{dt} = Px,$$

where t is a quantity on whose variation the motion of the point x depends. As in the case of two dimensions the most general proper orthogonal transformation is obtained by integrating this on the hypothesis that the elements of P are constant. Taking x for the initial and x' for the final position of the moving point, we can integrate this equation in the symbolic form

$$x' = e^{tP} x,$$

and it remains to express e^{tP} in a non-transcendental form. This may be done by a general method applicable to matrices of any order.†

For if the identical equation satisfied by a matrix P is

$$f(P) \equiv (P - \lambda_1 E)(P - \lambda_2 E) \cdots (P - \lambda_n E) = 0,$$

we have, by division, assuming the latent roots to be distinct,

$$P^m \equiv (P^{m-n} + \cdots) f(P) + \sum_{s=1}^{n} \frac{\lambda_s^m}{f'(\lambda_s)} \frac{f(P)}{P - \lambda_s E};$$

* Equivalent to rotations about a fixed point in four dimensions; from this point of view the subject has been treated by Cole, *Amer. Journ.*, vol. 12, p. 191, and Jahnke, *Jahresbericht der deutschen Math.-Vereinigung*, April, 1902.

† See Bromwich, *Proc. Camb. Phil. Soc.*, vol. 11, p. 75, and the references there given.

whence, using $f(P) = 0$,

$$\phi(P) = \sum_{s=1}^{n} \frac{\phi(\lambda_s)}{f'(\lambda_s)} \cdot \frac{f(P)}{P - \lambda_s E},$$

ϕ being any function expansible in ascending powers.* In the present instance the four-rowed matrix

$$P = \begin{bmatrix} 0 & -r & q & p' \\ r & 0 & -p & q' \\ -q & p & 0 & r' \\ -p' & -q' & -r' & 0 \end{bmatrix}$$

satisfies the equation

$$P^4 + (p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2) P^2 + (pp' + qq' + rr')^2 E = 0,$$

or say

$$(P^2 - \lambda^2 E)(P^2 - \mu^2 E) = 0.$$

Then, applying the rule given above,

$$e^P = \frac{e^{\lambda}}{2\lambda(\lambda^2 - \mu^2)} (P + \lambda E)(P^2 - \mu^2 E) + \frac{e^{-\lambda}}{-2\lambda(\lambda^2 - \mu^2)} (P - \lambda E)(P^2 - \mu^2 E) + \text{etc.}$$

$$= \frac{P^2 - \mu^2 E}{\lambda^2 - \mu^2} \left(\cosh \lambda E + \frac{1}{\lambda} \sinh \lambda P \right) + \frac{P^2 - \lambda^2 E}{\mu^2 - \lambda^2} \left(\cosh \mu E + \frac{1}{\mu} \sinh \mu P \right);$$

and this is the required matrix of the finite transformation after we have put $t = 1$.

11. It is evident that when the identical equation for P reduces to a binomial form, the evaluation of e^{tP} can be effected much more easily. This happens in two cases which will now be discussed, and the general case can be reduced to either of them.

* This formula was first given by Sylvester, *Johns Hopkins Univ. Circulars*, 3 (1882), pp. 9 and 210.

Let P' be the matrix obtained from P by interchanging p and p', q and q', r and r'. Then P and P' correspond to polar screws. It is easily verified by actual multiplication that

$$PP' = P'P = -(pp' + qq' + rr')E,$$

$$P^2 + P'^2 = -(p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2)E.$$

The first special case to be considered is when

$$P = P'.$$

Then P is called a *right-vector* and the identical equation takes the simple form

$$P^2 + (p^2 + q^2 + r^2)E = 0.$$

Putting

$$\theta = t\sqrt{p^2 + q^2 + r^2}$$

we have

$$e^{tP} = 1 + tP - \frac{\theta^2}{2} - \frac{t\theta^2}{3!}P + \cdots = \cos\theta + \frac{\sin\theta}{\theta}tP,$$

and in the corresponding displacement every point moves through the same distance θ along a right parallel of the system.*

Similarly if $P = -P'$, P is called a *left vector*, and the matrix of the displacement has the same form as before.

Now the general screw P can be expressed as the sum of a right vector and a left vector, for we have only to write

$$2U = P + P', \qquad 2V = P - P',$$

and then

$$P = U + V.$$

Further

$$UV = VU,$$

so that the matrices U, V, and therefore also the corresponding finite transformations, are commutative. Hence the general displacement e^{tP} can be resolved into the succession of vector displacements e^{tU}, e^{tV} taken in either order. Accordingly, if we put

* For a discussion of vector displacements see Whitehead, Universal Algebra, p. 472.

$$4\alpha^2 = (p+p')^2 + (q+q')^2 + (r+r')^2,$$

$$4\beta^2 = (p-p')^2 + (q-q')^2 + (r-r')^2,$$

$$U = \alpha U_1, \quad V = \beta V_1,$$

then

$$e^{tP} = (\cos \alpha t E + \sin \alpha t U_1)(\cos \beta t E + \sin \beta t V_1)$$

or

$$e^{tP} = \cos\alpha\cos\beta E + \sin\alpha\cos\beta U_1 + \cos\alpha\sin\beta V_1 + \sin\alpha\sin\beta U_1 V_1,$$

and this form may easily be made to agree with the preceding by putting

$$\alpha + \beta = i\lambda, \quad \alpha - \beta = i\mu.$$

12. The second special case is when

$$pp' + qq' + rr' = 0,$$

so that P represents a *line*. The corresponding displacement is a rotation about this line, and since $PP' = 0$ we deduce at once that rotations about polar lines are independent. Writing

$$c^2 = p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2,$$

we have

$$P^2 + P'^2 + c^2 E = 0,$$

and therefore

$$P^3 + c^2 P = 0.$$

Integration in this case is precisely the same as in two dimensions and we obtain for the finite equation of rotation through an angle θ

$$x' = e^{tP}x = \{1 + c^{-1}\sin\theta\, P + c^{-2}(1-\cos\theta)P^2\}x.$$

Now the general screw P can be expressed as the sum of two polar lines, its axes.* These are the lines L, L' where

$$L = \sigma P + \sigma^{-1}P', \quad L' = \sigma^{-1}P + \sigma P',$$

* Whitehead, Universal Algebra, p. 401.

and $LL' = 0$; giving

$$(\sigma^2 + \sigma^{-2})(pp' + qq' + rr') + c^2 = 0.$$

Write for abbreviation

$$\omega = pp' + qq' + rr',$$

then the sum of the squares of the six elements of L is

$$(\sigma^2 + \sigma^{-2})c^2 + 4\omega, \quad = -\omega(\sigma^2 - \sigma^{-2})^2;$$

accordingly it is convenient to introduce a new matrix M given by

$$L = (\sigma^2 - \sigma^{-2})\sqrt{-\omega}M.$$

Then

$$P = \frac{\sigma L - \sigma^{-1}L'}{\sigma^2 - \sigma^{-2}}$$

$$= \sigma\sqrt{-\omega}M - \sigma^{-1}\sqrt{-\omega}M'.$$

Accordingly the displacement defined by P is equivalent to rotations about the lines M, M', the angles being $\theta = \sigma\sqrt{-\omega}$ and $\theta' = -\sigma^{-1}\sqrt{-\omega}$. The matrix e^P of the finite transformation has the form

$$E + \sin\theta M + \sin\theta' M' + (1 - \cos\theta)M^2 + (1 - \cos\theta')M'^2,$$

which may be compared with either of the preceding forms.

13. As in the case of two dimensions, when $\theta \neq \pi$, $e^{\theta M}$ can be expressed in the form

$$\left(E + \tan\frac{\theta}{2}M\right)\left(E - \tan\frac{\theta}{2}M\right)^{-1}.$$

Hence, since $MM' = 0$,

$$e^P = e^{\theta M + \theta' M'} = e^{\theta M}e^{\theta' M'}$$

$$= \left(E + \tan\frac{\theta}{2}M + \tan\frac{\theta'}{2}M'\right)\left(E - \tan\frac{\theta}{2}M - \tan\frac{\theta'}{2}M'\right)^{-1},$$

so that in Cayley's expression for an orthogonal transformation the skew matrix used is

$$\tan\frac{\theta}{2}M + \tan\frac{\theta'}{2}M'.$$

This matrix is closely connected with Burnside's theorem that the general screw displacement can be resolved into two half turns about lines cutting the axes of the screw at right angles.* If P and Q are the matrices associated with any two lines whose shortest distances are θ, θ', and if M, M' are their common normals, it is easy to verify that†

$$PQ - QP = -M\sin\theta\cos\theta' + M'\sin\theta'\cos\theta;$$

on the other hand the succession of half turns is represented by the matrix

$$(E + 2P^2)(E + 2Q^2);$$

and it may left as an exercise in matrix notation‡ to verify that this is the same as

$$(PQ + P'Q')(QP + Q'P')^{-1},$$

and that this is

$$(E - \tan\theta M + \tan\theta' M')(E + \tan\theta M - \tan\theta' M')^{-1}$$

which, as was shown above, is the matrix of a displacement on a screw whose axes are M, M', the angles of rotation being -2θ, $+2\theta'$ respectively.§

14. We must now establish the connection between quaternions and the parametric representation of the general displacement.‖ Taking as before the skew matrix P of the displacement to be

$$\begin{vmatrix} 0 & -r & q & p' \\ r & 0 & -p & q' \\ -q & p & 0 & r' \\ -p' & -q' & -r' & 0 \end{vmatrix},$$

* *Messenger of Math.*, vol. 23, p. 19; *Proc. L. M. S.*, vol. 26, p. 33, where a simple geometric proof is given. A very short analytic proof can be given by taking the axes of the screw to be opposite edges of a tetrahedron of reference self-conjugate with respect to the absolute.

† The formulæ for common normals are given in the *Messenger of Math.*, vol. 32, p. 31.

‡ Hudson, *Messenger of Math.*, vol. 32, p. 51.

§ This theorem, treated geometrically, has formed the subject of papers by Wiener, *Leipziger Berichte*, vol. 42 (1890), Gale, *Annals of Math.*, Oct., 1900; Wood, *Annals of Math.*, July, 1901.

‖ The fact that a special four-rowed matrix is equivalent to a quaternion is due to Frobenius, *Crelle*, vol. 84, p. 59.

and putting

$$\alpha_0 = \cos\alpha, \quad \alpha_1 = \sin\alpha\frac{p+p'}{2\alpha}, \quad \alpha_2 = \sin\alpha\frac{q+q'}{2\alpha},$$

$$\alpha_3 = \sin\alpha\frac{r+r'}{2\alpha},$$

so that

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

the matrix of the right vector displacement

$$x' = \left(\cos\alpha\, E + \sin\alpha\frac{P+P'}{2\alpha}\right)x$$

is

$$\begin{bmatrix} \alpha_0 & -\alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_3 & \alpha_0 & -\alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 & \alpha_0 & \alpha_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \alpha_0 \end{bmatrix},$$

and that of the left vector displacement

$$x' = \left(\cos\beta E + \sin\beta\frac{P-P'}{2\beta}\right)x$$

is

$$\begin{bmatrix} \beta_0 & -\beta_3 & \beta_2 & -\beta_1 \\ \beta_3 & \beta_0 & -\beta_1 & -\beta_2 \\ -\beta_2 & \beta_1 & \beta_0 & -\beta_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_0 \end{bmatrix},$$

where

$$\beta_0 = \cos\beta, \quad \beta_1 = \sin\beta\frac{p-p'}{2\beta}, \quad \beta_2 = \sin\beta\frac{q-q'}{2\beta},$$

$$\beta_3 = \sin\beta\frac{r-r'}{2\beta},$$

and

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1.$$

Expressed in quaternion notation these transformations are

$$x_4' + ix_1' + jx_2' + kx_3' = (\alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3)(x_4 + ix_1 + jx_2 + kx_3)$$

and

$$x_4' + ix_1' + jx_2' + kx_3' = (x_4 + ix_1 + jx_2 + kx_3)(\beta_0 - i\beta_1 - j\beta_2 - k\beta_3)$$

respectively, which may be written

$$x' = ax \quad \text{and} \quad x' = xb^{-1},$$

all the letters representing unit quaternions. Combining these two vector displacements, we find for the general displacement either

$$x' = ax,\ x'' = x'b^{-1} \text{ giving } x'' = axb^{-1},$$

or

$$x' = xb^{-1},\ x'' = ax' \text{ also giving } x'' = axb^{-1},$$

showing once more that right and left vector displacements are commutative.*

Now put

$$v = ip + jq + kr, \quad v' = ip' + jq' + kr',$$

then the formula $x' = axb^{-1}$

is the same as

$$x' = \left(\cos\alpha + \sin\alpha\frac{v+v'}{2\alpha}\right)x\left(\cos\beta - \sin\beta\frac{v-v'}{2\beta}\right)$$

$$\text{or, } x' = \left(\frac{v+v'}{2\alpha}\right)^{\frac{2\alpha}{\pi}}(x)\left(\frac{v-v'}{2\beta}\right)^{-\frac{2\beta}{\pi}},$$

and it may be noticed that

$$2\alpha = T(v+v'), \quad 2\beta = T(v-v').$$

To pass to euclidean or hyperbolic space, replace x_4 by x_4/ω and v' by $\omega v'$, and ultimately put $\omega = 0$ or i respectively.† In

* Klein, Nicht-Euklidische Geometrie, II, p. 123.

† For further explanation see *Messenger of Math.*, vol. 31, p. 151, where the application of biquaternions is considered.

the former case we may at once neglect ω^2, and then the quaternion a takes the form $q + \omega q'$, where q is a unit quaternion and q' is connected with q only by the relation

$$Sq'q^{-1} = 0,$$

since $Ta = 1$. Evidently b takes the form $q - \omega q'$ and so, if we put

$$\xi = ix_1 + jx_2 + kx_3, \quad \xi' = ix_1' + jx_2' + kx_3',$$

the general displacement is given by

$$\frac{x_4'}{\omega} + \xi' = (q + \omega q')\left(\frac{x_4}{\omega} + \xi\right)(q - \omega q')^{-1}$$

$$= (q + \omega q')\left(\frac{x_4}{\omega} + \xi\right)(q^{-1} + \omega r),$$

where $qr - q'q^{-1} = 0$; and this is equivalent to

$$x_4' = x_4, \quad \xi' = q\xi q^{-1} + (qr + q'q^{-1})x_4,$$

$$= q\xi q^{-1} + 2q'q^{-1}x_4.$$

In this we may put $x_4 = 1$ and so obtain the general displacement in euclidean space in terms of 8 homogeneous parameters, the constituents of q and q', connected by the two relations *

$$Tq = 1, \quad Sq'q^{-1} = 0.$$

The ends set forth in the introduction have now been achieved.

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* Cf. Study, *Math. Annalen*, vol. 39, p. 536.

NOTES.

THE January number (second series, volume 4, number 2) of the *Annals of Mathematics* contains the following papers: "The logarithm as a direct function," by J. W. BRADSHAW, with an introduction by W. F. OSGOOD; "On positive quadratic forms," by PAUL SAUREL; "Multiple points on Lissajous's curves in two and three dimensions," by E. A. HOOK; "A special quadri-quadric transformation of real points in a plane," by C. C. ENGBERG.

The editorial staff of the *Annals* have recently added Dr. E. V. HUNTINGTON to their number.

AT the recent meeting of Section A of the American association for the advancement of science, held at Washington, D. C., twenty-three papers were read, of which the following related to pure mathematics: By Professor L. G. WELD: "Saint Loup's linkage."—By Dr. PAUL CARUS: "The foundations of mathematics."—By Professor G. B. HALSTED: "The teaching of geometry"; "The Bolyai centenary."—By Professor E. O. LOVETT: "Special periodic solutions of the problem of n bodies"; "The problems of three or more bodies with prescribed orbits."—By Mr. J. J. QUINN: "A development of conic sections by kinematic methods."—By Professor J. S. MILLER: "Note on a geometrical analysis" (by title).—By Dr. C. J. KEYSER: "Concerning Bolzano's contributions to assemblage theory" (by title).

AT the meeting of the London mathematical society held on January 8, the following papers were read: By Professor A. LODGE: "A method of representing imaginary points by real points in a plane."—By Dr. J. LARMOR: "On the mathematical expression of the principle of Huygens."—By Professor A. E. H. LOVE: "Wave motion with discontinuities at wave points."—By Dr. H. F. BAKER: "Functions of several variables."—By Mr. W. H. YOUNG: "On non-uniform convergence and term-by-term integration of series."—By Professor L. E. DICKSON: "Generational relations for the abstract group simply isomorphic with the linear fractional group in the Galois field $[2^n]$."—By Rev. F. H. JACKSON: "Series connected with

the enumeration of partitions (second paper)."—By Mr. J. BRILL: "On the minors of a skew-symmetrical determinant." —By Professor W. S. BURNSIDE: "On the Jacobian of two binary quantics considered geometrically"; "On the resolution of some skew invariants of binary quantics into their factors in terms of their roots."

At the meeting of November 13, Professor HORACE LAMB was elected president, and Professors A. E. H. LOVE and W. BURNSIDE were elected secretaries of the Society.

HARVARD UNIVERSITY: At the summer school, held in July and August, there will be offered for the first time an advanced mathematical course, the subject being: The theory of functions of a complex variable. The course is given by Professors OSGOOD and BÔCHER. Like the other courses given in the summer school, it is intended to occupy the students' entire working time during the six weeks of its continuance.

THE Macmillan Company announce the early publication of the following mathematical works: Elements of the theory of integers, by JOSEPH BOWDEN; The principles of mathematics, Volume 1, by BERTRAND RUSSELL; A treatise on determinants, by R. F. SCOTT, new edition edited by G. B. MATHEWS; The algebra of invariants, by J. H. GRACE and A. YOUNG; A treatise on spherical astronomy, by R. S. BALL. Longmans, Green and Company have in press: A first course in the infinitesimal calculus, by D. A. MURRAY.

THE Hungarian academy of sciences announces two mathematical prizes of 2000 crowns each, the first being offered for a manual of absolute geometry, of about 25 folios, to be submitted before December 31, 1904; the second for a study of a class of differential invariants, to be submitted before December 31, 1903.

THE annual register of the German Mathematiker-Vereinigung, issued in January, shows that forty-nine Americans are members of the society, thirteen having been elected during the year 1902.

PROFESSOR H. POINCARÉ has been made a commander of the Legion of honor.

PROFESSOR A. R. FORSYTH has been elected president of the Mathematical association.

THE College entrance examination board has appointed as examiners in mathematics for 1903 Professor CHARLOTTE A. SCOTT, Professor W. H. METZLER and Mr. J. S. FRENCH.

DR. O. D. KELLOGG has been appointed instructor in mathematics at Princeton University.

SIR GEORGE GABRIEL STOKES, master of Pembroke college, Cambridge, Lucasian professor of mathematics since 1849, died on February 1. He was born at Skreen, Ireland, in 1819, and was educated at Pembroke, taking his degree in 1841, being senior wrangler. He was president of the Royal society from 1885 to 1890, and from 1887 to 1892 he represented Cambridge University in parliament.

JAMES GLAISHER, the meteorologist, aëronaut and mathematician, died at London on February 8. He was born at London April 7, 1809. In 1829 he was employed in the ordnance survey of Ireland, in 1833 he became assistant at the Cambridge observatory, in 1836 he was appointed assistant in the Greenwich observatory, and in 1840 he became superintendent of the magnetical and meteorological department. He was a fellow of the Royal society (1849) and president of the Meteorological society, and was connected with various other learned associations. In mathematics he is chiefly known for his work on factor tables.

REV. H. W. WATSON, D.Sc., F.R.S., at one time mathematical lecturer at King's College, London, and Trinity College, Cambridge, died on January 11, aged seventy-five years.

LAWRENCE SLUTER BENSON, for many years an ardent expounder of perverse mathematical doctrine, died at an advanced age at Newark, N. J., January 27.

A SIXTH edition of his catalogue of mathematical models for use in advanced mathematical instruction, has just been issued by Martin Schilling, Halle a. S.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOHREN (A.). Ueber die Fresnelschen Integrale. Bern, 1901. 8vo. 48 pp.

ECHOLS (W. H.). An elementary text-book on the differential and integral calculus. New York, Holt, 1902. 8vo. 10 + 480 pp. Cloth. $2.00

FREGE (G.). Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet. Vol. II. Jena, Pohle, 1903. 8vo. 15 + 266 pp. M. 12.00

JECKLIN (L.). Historisch-kritische Untersuchung über die Theorie der hypergeometrischen Reihe bis zu den Entdeckungen von E. E. Kummer. Bern, 1901. 8vo. 87 pp.

JUNKER (F.). Höhere Analysis. Teil I: Differentialrechnung. 2te, verbesserte Auflage. 2ter Abdruck. Leipzig, Göschen, 1902. 12mo. 231 pp. Cloth. (Sammlung Göschen, No. 87.) M. 0.80

KREBS (A.). Konstruktionen gleichschenkliger Dreiecke mit Hilfe von Kurven höherer Ordnung. Bern, 1902. 8vo. 95 pp.

KUMMER (M.). Darlegung der Weberschen und verwandter Integrale, ihre Theorie und Anwendung. Bern, 1902. 8vo. 63 pp.

MACFARLANE (A.). A report on recent progress in the quaternion analysis. (*Proceedings of the American Association for the Advancement of Science*, Vol. 51, pp. 305–326. 1902.) 8vo.

NEWSON (H. B.). Report on the theory of collineations. (*Proceedings of the American Association for the Advancement of Science*, Vol. 51, pp. 579–599. 1902.) 8vo.

PALÁGYI (M.). Kant und Bolzano. Eine kritische Parallele. Halle, 1902. 8vo. 11 + 124 pp. M. 3.00

RÉPERTOIRE bibliographique des sciences mathématiques. 2e série: fiches 1101 à 1200. Paris, Gauthier-Villars, 1902. 18mo. Fr. 2.00

SALMON (G.). Traité de géométrie analytique (courbes planes), destiné à faire suite au traité des sections coniques. Ouvrage traduit de l'anglais par O. Chemin, et suivi d'une étude sur les points singuliers par G. Halphen. 2e tirage. Paris, Gauthier-Villars, 1903. 8vo. 19 + 667 pp. Fr. 12.00

SCHOUTEN (G.). Inleiding tot de studie der elliptische functien van Weierstrass. Delft, 1902. 8vo. 152 pp. 3 fl. 90c.

SITZUNGSBERICHTE der Berliner Mathematischen Gesellschaft. Herausgegeben vom Vorstand der Gesellschaft. 1ster Jahrgang. Leipzig, Teubner, 1902. 8vo. 4 + 66 pp. M. 2.40

SPIESS (O.). Die Grundbegriffe der Iterations-Rechnung. Basel, 1902. 8vo. 34 pp.

STAEBLE (F.). Untersuchung der Flächen, deren Krümmungslinien bei orthogonaler Projektion auf eine andere Fläche wieder Krümmungslinien werden. (Diss.) München, 1901. 8vo. 31 pp.

STOUFF (X.). Sur la première lettre arithmétique d'Hermite à Jacobi. (*Bulletin des sciences mathématiques*, (2) 26, 1902.) 8vo. 7 pp.

STUDY (E.). Geometrie der Dynamen. Die Zusammensetzung von Kräften und verwandte Gegenstände der Geometrie. Leipzig, Teubner, 1903. 8vo. 13 + 603 pp. M. 21.00

THUE (A.). Om en pseudomekanisk methode i geometrien. Kristiania, 1902. 8vo. 11 pp. 1 kr. 75 ö.

II. ELEMENTARY MATHEMATICS.

ADAM (V.). Taschenbuch der Logarithmen für Mittelschulen und höhere Bildungsanstalten. 31ste Auflage. Stereotypausgabe. Wien, Mejstrik, 1903. 4to. 10 + 100 pp. Cloth. M. 1.00

ASHTON (C. H.) and MARSH (W. R.). Five place logarithmic tables together with a four place table of natural functions. New York, Scribner, 1902. 12mo. 21 + 93 pp. Cloth. $0.60

BARBISCH (H.). See JAHNE (J.).

BLAINE (R. G.). Some quick and easy methods of calculating: a simple explanation of the theory and use of the slide-rule, logarithms, etc. 2d edition, revised and enlarged. London, Spon, 1902. 12mo. 164 pp. Cloth. 2s. 6d.

BOS (H.). Géométrie élémentaire. 2le édition. Paris, Hachette, 1903. 16mo. 288 pp. Fr. 2.00

COMAS Y MUNTANER (J.). Programa de nociones y ejercicios de aritmética y geometría. Santiago, Tipografía Galaica, 1902. 8vo. 16 pp. (Instituto general y técnico de Santiago.) Fr. 0.50

DEKKER (P.). See NIEMÖLLER (F.).

DESSENON (E.). Traité de trigonométrie rectiligne, à l'usage des élèves de seconde et de première (C et D), de mathématiques (A et B), et des candidats au baccalauréat et aux écoles. 3e édition. Paris, Nony, 1903. 8vo. 2 + 304 pp. Fr. 1.00

DIETRICHKEIT (O.). 7stellige Logarithmen und Antilogarithmen aller 4stelligen Zahlen und Mantissen von 1000–9999, bezw. 0000–9999, mit Rand-Index und Interpolations-Einrichtung für 4- bis 7stelliges Schnellrechnen. Berlin, Springer, 1903. 8vo. 64 pp. Cloth. M. 3.00

EYSSÉRIC et PASCAL. Eléments d'algèbre, avec des compléments, à l'usage des élèves de l'enseignement secondaire et des candidats aux baccalauréats. 21e édition, entièrement refondue, conforme aux programmes du 31 mai 1902. Paris, Delagrave, 1902. 12mo. 256 pp.

HABERL (J.). Lehrbuch der allgemeinen Arithmetik und Algebra. Zum Gebrauche für Oberrealschulen und verwandte Lehranstalten. 8te Auflage. Wien, Braumüller, 1902. 8vo. 6 + 387 pp. Cloth. M. 3.80

JAHNE (J.) und BARBISCH (H.). Leitfaden der Geometrie und des geometrischen Zeichnens für Mädchen-Bürgerschulen. 3te Stufe. Wien, Manz, 1902. 8vo. 46 pp., 7 plates. M. 1.20

KLEINSCHMIDT (E.). Leitfaden der Geometrie und des geometrischen Zeichnens für Mädchen-Bürgerschulen. Teil I: Ite Klasse. 3te Auflage. Inhaltlich unveränderter, nach der neuen Rechtschreibung hergestellter Abdruck der 2ten Auflage. Wien, Hölder, 1902. 8vo. 3 + 67 pp. Cloth. M. 1.00

KÖLTZSCH (A.). Grundzüge der Raumlehre. Ein Lern- und Uebungsbuch zum Gebrauche in Volksschulen, Fortbildungschulen, Präparanden-Anstalten und Mittelschulen. Heft I. Mit etwa 500 Uebungsaufgaben. 3te, verbesserte Auflage. Leipzig, Merseburger, 1903. 8vo. 78 pp. M. 0.60

LONDON University intermediate mathematics; guide to intermediate examinations in arts and science. 6th edition. London, Clive, 1902. 8vo. 120 pp. 2s.

MILNE (W. J.). Advanced algebra for colleges and schools. New York, American Book Co., [1902.] 12mo. 608 pp. Leather. $1.50

MOLENBROEK (P.). Leerboek der meetkunde. 2ter druk. Deel I en II: Planimetrie; Stereometrie. Leiden, 1902. 8vo. 322 + 211 pp. M. 4.80

NIEMÖLLER (F.) und DEKKER (P.). Arithmetisches und algebraisches Unterrichtsbuch. Für den mathematischen Unterricht in der Mittel- und Oberstufe höherer Lehranstalten nach den Bestimmungen der neuesten preussischen Lehrpläne bearbeitet. (In 4 Heften.) Heft 3: Pensum der Obersekunda und der beiden Primen des Gymnasiums. Neue Auflage. Breslau, Hirt, 1902. 8vo. 96 pp. Boards. M. 1.40

PASCAL. See EYSSÉRIC.

PESCH (A. J. VAN). Logarithmentafels met 5 decimalen. 6ter druk. Leiden, 1902. 8vo. 22 + 96 pp. M. 2.00

PONS Y MERI (B.). Programa de aritmética. Valladolid, Montero, 1902. 4to. 20 pp. (Instituto general y técnico de Valladolid.) Fr. 1.00

——. Programa de geometría. Valladolid, Montero, 1902. 4to. 30 pp. (Instituto general y técnico de Valladolid.) Fr. 1.00

RAWLINS (J. M.). Key to Lippincott's Elementary algebra, in which are given solutions in full or in part to the more difficult exercises and problems. Philadelphia, Lippincott, [1902.] 12mo. 5 + 192 pp. Cloth. $1.00

SALOMON (A.). Leçons d'arithmétique, à l'usage de l'enseignement secondaire des jeunes filles (classes de deuxième et troisième années). 3e édition. Paris, Nony, 1903. 18mo. 8 + 223 pp. Fr. 2.00

——. Leçons de géométrie, à l'usage de l'enseignement secondaire des jeunes filles. Géométrie plane (classes de troisième et quatrième années). Paris, Nony, 1903. 18mo. 230 pp. Fr. 2.00

SCHULTZ (E.). Ausführlicher Leitfaden der Körperberechnung mit Musterbeispielen und einer Aufgabensammlung für gewerbliche Lehranstalten, sowie zum Selbstunterricht für den Maschinentechniker. 2te Auflage. Essen, Baedeker, 1903. 8vo. 4 + 192 pp. Boards. M. 2.00

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